GRADED MONOIDAL CATEGORIES AND INTERNALIZATION

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Abstract. In stable homotopy theory there are a variety of different ways to construct a point-set level smash product of spectra, each with its own advantages and limitations. Several of these constructions involve "internalizing" some kind of external product. In this paper, we aim to formalize this process which allows a graded, external product to be molded into an internal one. In this paper we introduce the general theory of internalization, investigate its categorical properties, and conclude by describing how both Day convolution and the smash product of S-modules fit into this framework.

Contents

1. Introduction

Unsurprisingly, in homotopy theory one is often concerned only with objects of interest up to homotopy. However, when possible, one may wish to lift a homotopical phenomenon to the point-set level, and study things prior to passing to the homotopy category. In the particular case of spectra, one might desire a well-behaved smash product which lifts the canonical one dwelling within the homotopy category of spectra. However, in [7], it is shown that no such smash product can exist, without surrendering at least one property desirable of this product.

In fact, many such weaker smash products of spectra exist, depending on which properties are relinquished and what point-set model of spectra is favored. For instance, one can form a smash product of symmetric or orthogonal spectra using Day convolution and the results of [8]. Alternatively, if one wishes to use the coordinate-free spectra of [1] then restricting to S-modules yields another smash product. Both of these constructions entail some "external" graded product being pulled into a single degree to yield the desired "internal" smash product. In this paper, we propose a general categorical axiomatization of this procedure of "internalizing" a graded product.

We begin by briefly outlining the general notational conventions used throughout this paper in Section 2. In Section 3 we introduce the basic definitions and results underpinning internalization: Section 3.1 is devoted to describing and providing a handful of examples of graded monoidal categories, while Section 3.2 goes on to define internalizations of these categories, before proving some fundamental results about internalization. In Section 4 we form the category of graded monoidal categories with internalizations by defining functors between them, and investigate some of the general categorical properties of internalization. Finally, we conclude with an overview of the two main examples of internalization in Section 5: Day convolution is reinterpreted through the lens of internalization in Section 5.1, and in Section 5.2 we reconstruct the smash product of S-modules as the internalization of the external product of spectra.

2. Notation

Before we begin assembling the machinery of internalization, we make a few brief notes on the notation used throughout this paper.

We denote by $\mathbb{1}$ the terminal category, consisting of a unique object and its identity morphism. Given a category $\mathcal{C}$ and objects $x, y \in \mathcal{C}$, we write $\mathcal{C}(x, y)$ for the set[1] of morphisms with domain x and codomain y. If $\mathcal{C}$ is instead enriched over some monoidal category $\mathcal{V}$, then in most cases we will write $\underline{\mathcal{C}}(x, y)$ for the hom-object in $\mathcal{V}$ of morphisms from x to y, to emphasize the enrichment.

Additionally, we will write $\mathbb{N}$ for the set of natural numbers (starting at 0). Given $m \in \mathbb{N}$ we denote by $\mathcal{C}^m$ the m^{th} power of $\mathcal{C}$, that is, the iterated product

$$\mathcal{C}^m := \prod_{i=1}^{m} \mathcal{C}.$$

Note that $\mathcal{C}^0 = \mathbb{1}$.

Finally, much of the content of the following sections involves objects which are $\mathbb{N}$-graded. Consequently, a great deal of indices are required to keep track of degrees. When possible, we omit these indices, provided that the degree of the objects we are concerned with are either clear from context or unimportant. In particular, our diagrams will (mercifully) almost always be free of indices.

3. External and Internal Products

Our aim is to formalize the procedure of coercing a kind of external, graded product into an ordinary monoidal product. To achieve this, we must first make sense of what it means for a (graded) category to be equipped with such an external product. We define the notion of a "graded monoidal product" in Section 3.1, and introduce generalizations of a few common properties enjoyed by ordinary monoidal categories (namely, symmetry and closure). We then go on to make rigorous the structure which enables internalizing a graded product, and establish some basic results regarding internalization, in Section 3.2.

Remark 3.1. All of the definitions and constructions in Section 3.1 and Section 3.2 can be interpreted more generally in the enriched setting. In particular, we could fix a

[1] For simplicity, we assume foundations that allow for all categories considered to be made locally small. The devout reader may substitute "class" for "set" if needed.

finitely-complete[2] closed monoidal category $(\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}})$ and consider external products (as $\mathcal{V}$-functors) on graded $\mathcal{V}$-categories. We omit a technical description of this generalization, as the interpretations of the arguments above would be at best cumbersome, and at worst entirely unreadable.

In Section 5.1 the internalized categories we are interested in will be enriched, so we note that this is possible only so that the reader is not completely blindsided by the sudden use of enriched categories.

3.1. **Graded Monoidal Categories.** We begin by making the following definition:

Definition 3.2. A *graded monoidal category* $(\mathcal{C}_\bullet, \mu, 1)$ consists of the following data:

(1) For each $m \in \mathbb{N}$ a category $\mathcal{C}_m$, called the *degree m component* of $\mathcal{C}_\bullet$. (We will also occasionally write $\mathcal{C}$ for the coproduct of the components of $\mathcal{C}_\bullet$.)
(2) A bifunctor $\mu_{m,n} : \mathcal{C}_m \times \mathcal{C}_n \to \mathcal{C}_{m+n}$ for each pair $m, n \in \mathbb{N}$. We will often abuse notation by writing μ for the gestalt of these functors. (We will often omit degree indices; since μ is degree-additive, the degree of $\mu(x, y)$ is uniquely determined by the degrees of x and y, so this is unambiguous.)
(3) An object $1 \in \mathcal{C}_0$, called the *unit* of μ.
(4) A natural isomorphism $\alpha_{m,n,r}$ for each $m, n, r \in \mathbb{N}$, called the *associators* of μ:

$$\begin{tikzcd} \mathcal{C}_m \times \mathcal{C}_n \times \mathcal{C}_r \arrow[r, "\mathcal{C}_m \times \mu_{n,r}"] \arrow[d, "\mu_{m,n} \times \mathcal{C}_r"'] \arrow[dr, phantom, "\alpha_{m,n,r} \swarrow"] & \mathcal{C}_m \times \mathcal{C}_{n+r} \arrow[d, "\mu_{m,n+r}"] \\ \mathcal{C}_{m+n} \times \mathcal{C}_r \arrow[r, "\mu_{m+n,r}"'] & \mathcal{C}_{m+n+r} \end{tikzcd} .$$

(5) Natural isomorphisms λ_m and ρ_m for each $m \in \mathbb{N}$, called the *left unitors* and *right unitors* of μ, respectively:

$$\begin{tikzcd} & \mathcal{C}_0 \times \mathcal{C}_m, \arrow[dr, "\mu_{0,m}"] \arrow[d, phantom, "\Downarrow \lambda_m"] & \\ \mathcal{C}_m \arrow[ur, "(1, \mathcal{C}_m)"] \arrow[rr, equal] & {} & \mathcal{C}_m \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}_m \arrow[rr, equal] \arrow[dr, "(\mathcal{C}_m, 1)"'] & {} \arrow[d, phantom, "\Uparrow \rho_m"] & \mathcal{C}_m \\ & \mathcal{C}_m \times \mathcal{C}_0 \arrow[ur, "\mu_{m,0}"'] & \end{tikzcd}$$

This data is further subject to the same coherence conditions as ordinary monoidal categories[3]; that is, for each $x, y, z, w \in \mathcal{C}$ the following diagrams commute:

$$\begin{tikzcd} & \mu(x, y) & \\ \mu(x, \mu(1, y)) \arrow[ur, "\mu(x, \lambda_y)"] \arrow[rr, "\alpha_{x,1,y}"'] & & \mu(\mu(x, 1), y) \arrow[ul, "\mu(\rho_x, y)"'] \end{tikzcd} \tag{3.3}$$

[2]The completeness hypothesis is necessary to form the binary product of $\mathcal{V}$-categories.

[3]If we were taking care to track degrees everywhere, the coherence conditions for graded monoidal categories would differ only in that they would be decorated with a cumbersome quantity of indices. In an effort to minimize the reader's vertigo, we will omit the degree indices from all diagrams throughout this paper.

$$
\begin{tikzcd}
\mu(x,\mu(y,\mu(z,w))) \arrow[rr, "\alpha_{x,y,\mu(z,w)}"] \arrow[d, "\mu(x,\alpha_{y,z,w})"'] & & \mu(\mu(x,y),\mu(z,w)) \arrow[d, "\alpha_{\mu(x,y),z,w}"] \\
\mu(x,\mu(\mu(y,z),w)) \arrow[dr, "\alpha_{x,\mu(y,z),w}"'] & & \mu(\mu(\mu(x,y),z),w) \\
& \mu(\mu(x,\mu(y,z)),w) \arrow[ur, "\mu(\alpha_{x,y,z},w)"'] &
\end{tikzcd}
\tag{3.4}
$$

We think of the collection of categories $\mathcal{C}_\bullet$ as a single monoidal category, with the additional structure of a grading. Indeed, μ defines an ordinary monoidal structure on $\mathcal{C} := \coprod_{m\in\mathbb{N}} \mathcal{C}_m$, since each of this category's morphisms are contained within its independent graded components (so the naturality and coherence of of the structure maps can be checked degree-wise).

Remark 3.5. It makes sense to allow more general notions of grading; a priori, Definition 3.2 makes sense for any collection of categories graded by a monoid $(M,\cdot,e)$, where the monoidal product has signature $\mu : \mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_{x\cdot y}$ and the identity $1^{\mathcal{C}}$ lives in $\mathcal{C}_e$. Naturally, one can further generalize to grading by a monoidal *category*. For reasons that will become apparent in Section 3 we are specifically interested in the case when the structure indexing a graded monoidal category is generated by a distinguished element (e.g. $1 \in \mathbb{N}$); for simplicity, we restrict ourselves to grading all of our categories by $\mathbb{N}$.

Examples of graded monoidal categories are abundant in nature:

Example 3.6. Any monoidal category $\mathcal{C}$ can be viewed as a graded monoidal category in a number of ways. One can form the *trivial graded category* $\mathcal{C}(0)_\bullet$, which is given by $\mathcal{C}$ in degree 0, and which is the empty category in all higher degrees. (One could equally take $\mathcal{C}(0)_m$ to be the terminal category $\mathbb{1}$ with one object and one morphism for each $m > 0$.) The monoidal product on $\mathcal{C}$ then naturally extends to a graded monoidal product on $\mathcal{C}(0)_\bullet$.

The ordinary monoidal category $\mathcal{C}$ also gives rise to the *constant graded category* $\overline{\mathcal{C}}_\bullet$, defined by $\overline{\mathcal{C}}_m := \mathcal{C}$ for each $m \in \mathbb{N}$. Again, it is clear that the monoidal product on $\mathcal{C}$ extends to a graded monoidal product on $\overline{\mathcal{C}}_\bullet$.

Finally, one can produce a *free graded monoidal category* $\mathrm{Free}(\mathcal{C})_\bullet$ on any given category $\mathcal{C}$, monoidal or not. We let $\mathrm{Free}(\mathcal{C})_m := \mathcal{C}^m$ for each $m \in \mathbb{N}$, where $\mathcal{C}^0 = \mathbb{1}$, and let $\mathrm{Free}(\mathcal{C})_m \times \mathrm{Free}(\mathcal{C})_n \to \mathrm{Free}(\mathcal{C})_{m+n}$ be given by concatenation. This construction enjoys a handful of nice properties when $\mathcal{C}$ is monoidal, which are outlined in Section 4 (see **??** and **??**).

Example 3.7. Every graded integral domain $S = \bigoplus_{d\geq 0} S_d$ forms a *preadditive graded monoidal category* $\mathcal{S}_\bullet$ – that is, a graded monoidal category such that $\mathcal{S}_m$ is enriched over abelian groups for each $m \in \mathbb{N}$. In each degree $\mathcal{S}_\bullet$ has only one object, while the morphisms of $\mathcal{S}_m$ are those elements of S with degree m (i.e. which have nonzero projections to S_m, and trivial projections to all higher degrees). The addition of morphisms within the same component is defined using the addition of S. In fact, we can further define the sum of morphisms in different components this way, since the degree of an element of S is well-defined, so $\mathcal{S} := \coprod_{m\in\mathbb{N}} \mathcal{S}_m$ is itself a preadditive category.

We equip $\mathcal{S}_\bullet$ with the graded product $\mathcal{S}_m \times \mathcal{S}_n \to \mathcal{S}_{m+n}$ which acts on morphisms via the multiplication of S. Since S is a graded ring, and the product of elements of S of degrees m

and n is an element of degree $m+n$ (hence the requirement that S be an integral domain), this defines a graded monoidal product. We note that the set of morphisms in $\mathcal{S}$ corresponds precisely to the underlying set of S.

Example 3.8. The category **FinMan** of finite-dimensional (topological, n-differentiable, smooth, etc.) connected manifolds can be turned into a graded category $\mathbf{FinMan}_\bullet$, where $\mathbf{FinMan}_m$ is the full subcategory of m-dimensional manifolds. The categorical product of **FinMan** then forms a graded monoidal product on $\mathbf{FinMan}_\bullet$, since it is dimension-additive.

Note that we have lost the data of morphisms between manifolds of distinct dimension in forming $\mathbf{FinMan}_\bullet$. As we will see in the following sections, the purpose of a graded monoidal category will often be to study the degree 1 component in particular. In practice, we will typically build the higher degrees of a graded monoidal category from degree 1, as opposed to stripping a category of enough data to reduce it to a graded monoidal category.

As a kind of specialization of monoidal categories, we would like to extend some of the properties desired of an ordinary monoidal product to the graded setting. We start by describing what it means for a graded monoidal product to be symmetric. Throughout the remainder of this section, we fix a graded monoidal category $(\mathcal{C}_\bullet, \mu, 1)$, using the same notation for the associators and unitors as above.

Definition 3.9. For each $m, n \in \mathbb{N}$, we write $\tau_{m,n}$ for the *transposition functor* $\mathcal{C}_m \times \mathcal{C}_n \to \mathcal{C}_n \times \mathcal{C}_m$. Explicitly, this is defined via the universal property of the product to be the unique functor making

$$\begin{tikzcd} & \mathcal{C}_m \times \mathcal{C}_n \arrow[dl] \arrow[d, dashed, "\tau_{m,n}"] \arrow[dr] & \\ \mathcal{C}_n & \mathcal{C}_n \times \mathcal{C}_m \arrow[l] \arrow[r] & \mathcal{C}_m \end{tikzcd}$$

commute, where the unlabeled arrows are the appropriate projection functors.

Definition 3.10. $\mathcal{C}_\bullet$ is *symmetric* if, for each $m, n \in \mathbb{N}$, there exist natural isomorphisms $\sigma_{m,n} : \mu_{m,n} \cong \mu_{n,m}\tau_{m,n}$, called the *braidings* of μ, making the following diagrams commute for each $x, y, z \in \mathcal{C}$:

$$\begin{tikzcd} & x & \\ \mu(1,x) \arrow[ur, "\lambda_x"] \arrow[rr, "\sigma_{1,x}"'] & & \mu(x,1) \arrow[ul, "\rho_x"'] \end{tikzcd} \tag{3.11}$$

$$\begin{tikzcd} \mu(x,\mu(y,z)) \arrow[r, "\mu(x,\sigma_{y,z})"] \arrow[d, "\alpha_{x,y,z}"'] & \mu(x,\mu(z,y)) \arrow[d, "\alpha_{x,z,y}"] \\ \mu(\mu(x,y),z) \arrow[d, "\sigma_{\mu(x,y),z}"'] & \mu(\mu(x,z),y) \arrow[d, "\mu(\sigma_{x,z},y)"] \\ \mu(z,\mu(x,y)) \arrow[r, "\alpha_{z,x,y}"'] & \mu(\mu(z,x),y) \end{tikzcd} \tag{3.12}$$

$$
\begin{tikzcd}
\mu(x,y) \arrow[rr, equal] \arrow[dr, "\sigma_{x,y}"'] & & \mu(x,y) \\
& \mu(y,x) \arrow[ur, "\sigma_{y,x}"'] &
\end{tikzcd}
\tag{3.13}
$$

As in Definition 3.2, this definition directly generalizes the classical case. The coherence conditions for graded symmetric monoidal categories are precisely the same as the usual ones, with the exception that the sufficiently motivated reader could insert degree indices in each diagram above. We can likewise stratify the notion of a closed monoidal category by grading the internal hom:

Definition 3.14. $\mathcal{C}_\bullet$ is *closed* if, for each $m, n \in \mathbb{N}$, there exists a bifunctor $\mathscr{F}_{m,n} : \mathcal{C}_n^{\mathrm{op}} \times \mathcal{C}_{m+n} \to \mathcal{C}_m$ together with a natural isomorphism

$$
\begin{tikzcd}
\mathcal{C}_m^{\mathrm{op}} \times \mathcal{C}_n^{\mathrm{op}} \times \mathcal{C}_{m+n} \arrow[r, "\mu_{m,n} \times \mathcal{C}_{m+n}"] \arrow[d, "\mathcal{C}_m^{\mathrm{op}} \times \mathscr{F}_{m,n}"'] \arrow[dr, phantom, "\overset{\sim}{\Longleftarrow}"] & \mathcal{C}_{m+n}^{\mathrm{op}} \times \mathcal{C}_{m+n} \arrow[d, "\mathcal{C}_{m+n}(-,-)"] \\
\mathcal{C}_m^{\mathrm{op}} \times \mathcal{C}_m \arrow[r, "\mathcal{C}_m(-,-)"'] & \mathbf{Set}\ .
\end{tikzcd}
$$

The collection of bifunctors $\mathscr{F}$ are called the *internal hom* of $\mathcal{C}_\bullet$.

Remark 3.15. In the case that μ is symmetric, $\mathscr{F}$ defines a two-variable adjunction with μ.

Remark 3.16. If concerned with the $\mathcal{V}$-enriched setting, as outlined in Remark 3.1, one simply replaces the hom-set functors $\mathcal{C}_k(-,-) : \mathcal{C}_k^{\mathrm{op}} \times \mathcal{C}_k \to \mathbf{Set}$ in Definition 3.14 with the $\mathcal{V}$-enriched hom-functors $\underline{\mathcal{C}_k}(-,-) : \mathcal{C}_k^{\mathrm{op}} \times \mathcal{C}_k \to \mathcal{V}$.

Remark 3.17 (Coherence)**.** As noted previously, everything defined above is a direct extension of ordinary monoidal categories. All of the coherence conditions are exactly the same as usual, ignoring the minor technicality of tracking degrees. Consequently, it should not be surprising that Mac Lane's Coherence Theorem for monoidal categories [6, Section VII] will still hold in the slightly more general setting of graded monoidal categories.

The main idea is that the usual proof of the theorem can be adapted by formally inserting degree indices where necessary, when constructing the preorder category of words as in VII.2 of [6]. At the beginning of the proof one must exercise slightly more caution when assembling each set of valid words inductively, but ultimately this does not cause any issues, and the rest of the argument is identical. We will not need the coherence theorem in the sequel, but it is worth mentioning, if only to emphasize that graded monoidal categories are not so far a departure from their ungraded cousins.

3.2. **Internalization.** In the preceding section, we laid the foundations for studying graded monoidal products. In this section, we describe the additional structure required to drag the external product of a graded monoidal category into the degree one component[4], and explore some of the basic behavior of this structure.

Definition 3.18. Let $(\mathcal{C}_\bullet, \mu, 1)$ be a graded monoidal category. An *internalization of* μ consists of:

[4]There is nothing exceptional about degree one, and in theory one could define an internalization into any degree. However, in practice we are most concerned with the degree one component, so we elect to bake our bias towards this component into our definitions for simplicity.

(1) For each $m \in \mathbb{N}$ a functor $\Phi_m : \mathcal{C}_m \to \mathcal{C}_1$, with a natural isomorphism ι which identifies Φ_1 with the identity functor:

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "\Phi_1"{name=U}] \arrow[r, bend right, equal, ""'{name=D}] & \mathcal{C}_1 \arrow[Rightarrow, from=U, to=D, "\iota"] \end{tikzcd} .$$

(As usual, we will write Φ to denote the collection of all the Φ_m).

(2) For each $m, n \in \mathbb{N}$, natural isomorphisms $\mathcal{L}_{m,n}$ and $\mathcal{R}_{m,n}$, called the *left absorbers* and *right absorbers* of Φ, respectively[5]:

$$\begin{tikzcd}
 & \mathcal{C}_1 \times \mathcal{C}_n \arrow[rr, "\mu_{1,n}"] & {} \arrow[d, phantom, "\Downarrow\mathcal{L}_{m,n}"] & \mathcal{C}_{n+1} \arrow[dr, "\Phi_{n+1}"] & \\
\mathcal{C}_m \times \mathcal{C}_n \arrow[ur, "\Phi_m \times \mathcal{C}_n"] \arrow[rr, "\mu_{m,n}"] \arrow[dr, "\mathcal{C}_m \times \Phi_n"'] & & \mathcal{C}_{m+n} \arrow[rr, "\Phi_{m+n}"] & & \mathcal{C}_1 \\
 & \mathcal{C}_m \times \mathcal{C}_1 \arrow[rr, "\mu_{m,1}"'] & {} \arrow[u, phantom, "\Uparrow\mathcal{R}_{m,n}"] & \mathcal{C}_{m+1} \arrow[ur, "\Phi_{m+1}"'] &
\end{tikzcd} .$$

This data is further subject to the coherence conditions below – given $x, y, z \in \mathcal{C}$, the following diagrams commute:

$$\begin{tikzcd}
\Phi\mu(\Phi x, \Phi y) \arrow[r, "\mathcal{L}_{x,\Phi y}"] \arrow[d, "\mathcal{R}_{\Phi x, y}"'] & \Phi\mu(x, \Phi y) \arrow[d, "\mathcal{R}_{x,y}"] \\
\Phi\mu(\Phi x, y) \arrow[r, "\mathcal{L}_{x,y}"'] & \Phi\mu(x, y),
\end{tikzcd} \tag{3.19}$$

$$\begin{tikzcd}
\Phi\mu(x, \Phi\mu(y, \Phi z)) \arrow[r, "\mathcal{R}_{x,\mu(y,\Phi z)}"] \arrow[d, "\Phi\mu(x, \mathcal{R}_{y,z})"'] & \Phi\mu(x, \mu(y, \Phi z)) \arrow[r, "\Phi\alpha_{x,y,\Phi z}"] & \Phi\mu(\mu(x, y), \Phi z) \arrow[d, "\mathcal{R}_{\mu(x,y),z}"] \\
\Phi\mu(x, \Phi\mu(y, z)) \arrow[r, "\mathcal{R}_{x,\mu(y,z)}"'] & \Phi\mu(x, \mu(y, z)) \arrow[r, "\Phi\alpha_{x,y,z}"'] & \Phi\mu(\mu(x, y), z),
\end{tikzcd} \tag{3.20}$$

$$\begin{tikzcd}
\Phi\mu(\Phi\mu(\Phi x, y), z) \arrow[r, "\mathcal{L}_{\mu(\Phi x,y),z}"] \arrow[d, "\Phi\mu(\mathcal{L}_{x,y}, z)"'] & \Phi\mu(\mu(\Phi x, y), z) \arrow[r, "\Phi\alpha^{-1}_{\Phi x,y,z}"] & \Phi\mu(\Phi x, \mu(y, z)) \arrow[d, "\mathcal{L}_{x,\mu(y,z)}"] \\
\Phi\mu(\Phi\mu(x, y), z) \arrow[r, "\mathcal{L}_{\mu(x,y),z}"'] & \Phi\mu(\mu(x, y), z) \arrow[r, "\Phi\alpha_{x,y,z}"'] & \Phi\mu(x, \mu(y, z)),
\end{tikzcd} \tag{3.21}$$

$$\begin{tikzcd}
\Phi\mu(x, \Phi\mu(y, z)) \arrow[r, "\mathcal{R}_{x,\mu(y,z)}"] \arrow[d, "\Phi\mu(x, \mathcal{L}^{-1}_{y,z})"'] & \Phi\mu(x, \mu(y, z)) \arrow[r, "\Phi\alpha_{x,y,z}"] & \Phi\mu(\mu(x, y), z) \arrow[d, "\mathcal{L}^{-1}_{\mu(x,y),z}"] \\
\Phi\mu(x, \Phi\mu(\Phi y, z)) \arrow[d, "\mathcal{R}_{x,\mu(\Phi y,z)}"'] & & \Phi\mu(\Phi\mu(x, y), z) \arrow[d, "\Phi\mu(\mathcal{R}^{-1}_{x,y}, z)"] \\
\Phi\mu(x, \mu(\Phi y, z)) \arrow[r, "\Phi\alpha_{x,\Phi y,z}"'] & \Phi\mu(\mu(x, \Phi y), z) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,\Phi y),z}"'] & \Phi\mu(\Phi\mu(x, \Phi y), z),
\end{tikzcd} \tag{3.22}$$

[5]The absorbers are so-named because they allow the outermost factor of Φ to "absorb" the inner ones.

and lastly that

$$\mathcal{R}_{m,1} = \Phi_{m+1}\mu(\mathrm{id}, \iota) \tag{3.23}$$

$$\mathcal{L}_{1,m} = \Phi_{m+1}\mu(\iota, \mathrm{id}). \tag{3.24}$$

If $(\mathcal{C}_\bullet, \mu, 1)$ is equipped with an internalization Φ as above, we say that $\mathcal{C}_\bullet$ is an *internalized monoidal category* (we omit the term "graded" for brevity – occasionally we will refer to these as "internalized categories"), and write $(\mathcal{C}_\bullet, \Phi)$ when Φ is not clear from context, leaving the absorbers implicit.

As suggested by the name, the purpose of an internalization Φ on a graded monoidal category $\mathcal{C}_\bullet$ is to "internalize" its external product μ. The functors Φ allow us to normalize the degree of μ, while the absorbers enable us to extract the ordinary monoidal coherence axioms from the graded ones.

Theorem 3.25. Let $(\mathcal{C}_\bullet, \mu, 1)$ be a graded monoidal category equipped with an internalization $(\Phi, \mathcal{L}, \mathcal{R})$. Then the bifunctor $\otimes : \mathcal{C}_1 \times \mathcal{C}_1 \to \mathcal{C}_1$ defined by

$$x \otimes y := \Phi_2\mu_{1,1}(x, y)$$

defines a monoidal product on $\mathcal{C}_1$, with:

(1) Unit given by $I := \Phi_0(1)$.
(2) associator a given by the composite

$$a_{x,y,z} := (\mathcal{L}^{-1}_{2,1})_{\mu(x,y),z} \circ \Phi_3(\alpha_{1,1,1})_{x,y,z} \circ (\mathcal{R}_{1,2})_{x,\mu(y,z)},$$

for each $x, y, z \in \mathcal{C}_1$, depicted by the pasting diagram below:

$$\begin{tikzcd}
 & \mathcal{C}_1 \times \mathcal{C}_2 \arrow[r, "\mathcal{C}_1 \times \Phi_2"] \arrow[dr, "\mu_{1,2}"] \arrow[dd, phantom, "\Downarrow \alpha_{1,1,1}"] & \mathcal{C}_1 \times \mathcal{C}_1 \arrow[r, "\mu_{1,1}"] & \mathcal{C}_2 \arrow[dr, "\Phi_2"] \arrow[d, phantom, "\swarrow \mathcal{R}_{1,2}"] & \\
\mathcal{C}_1 \times \mathcal{C}_1 \times \mathcal{C}_1 \arrow[ur, "\mathcal{C}_1 \times \mu_{1,1}"] \arrow[dr, "\mu_{1,1} \times \mathcal{C}_1"'] & & \mathcal{C}_3 \arrow[rr, "\Phi_3"] & \phantom{\mathcal{C}_2} \arrow[d, phantom, "\swarrow \mathcal{L}^{-1}_{2,1}"] & \mathcal{C}_1 \\
 & \mathcal{C}_2 \times \mathcal{C}_1 \arrow[r, "\Phi_2 \times \mathcal{C}_1"'] \arrow[ur, "\mu_{2,1}"'] & \mathcal{C}_1 \times \mathcal{C}_1 \arrow[r, "\mu_{1,1}"'] & \mathcal{C}_2 \arrow[ur, "\Phi_2"'] & .
\end{tikzcd}$$

(3) left and right unitors l and r (respectively), given by

$$l_x := \iota_x \circ \Phi_1(\lambda_1)_x \circ (\mathcal{L}_{0,1})_{1,x}$$
$$r_x := \iota_x \circ \Phi_1(\rho_2)_x \circ (\mathcal{R}_{1,0})_{x,1},$$

for each $x \in \mathcal{C}_1$, illustrated by the following pasting diagrams:

$$\begin{tikzcd}
 & \mathcal{C}_0 \times \mathcal{C}_1 \arrow[r, "\Phi_0 \times \mathcal{C}_1"] \arrow[dr, "\mu_{0,1}"] \arrow[d, phantom, "\Downarrow \lambda_1"] & \mathcal{C}_1 \times \mathcal{C}_1 \arrow[r, "\mu_{1,1}"] \arrow[dr, phantom, "\swarrow \mathcal{L}_{0,1}"] & \mathcal{C}_2 \arrow[d, "\Phi_2"] \\
\mathcal{C}_1 \arrow[ur, "(1,\mathcal{C}_1)"] \arrow[rr, equal] \arrow[rrr, equal, bend right=30] \arrow[rrr, phantom, bend right=15, "\Downarrow \iota"] & & \mathcal{C}_1 \arrow[r, "\Phi_1"'] & \mathcal{C}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_1 \arrow[rrr, equal, bend left=30] \arrow[rrr, phantom, bend left=15, "\iota \Uparrow"] \arrow[dr, phantom, "\mathcal{R}_{1,0} \nearrow"] & \mathcal{C}_1 \arrow[l, "\Phi_1"'] \arrow[rr, equal] & & \mathcal{C}_1 \arrow[dl, "(\mathcal{C}_1,1)"] \\
\mathcal{C}_2 \arrow[u, "\Phi_2"] & \mathcal{C}_1 \times \mathcal{C}_1 \arrow[l, "\Phi_1"] & \mathcal{C}_1 \times \mathcal{C}_0 \arrow[l, "\mathcal{C}_1 \times \Phi_0"] \arrow[ul, "\mu_{1,0}"] \arrow[ur, phantom, "\rho_1 \Uparrow"] & .
\end{tikzcd}$$

We say that $\otimes$ is the *internalization of* μ *with respect to* Φ.

Proof. We must verify that the specified natural transformations satisfy the coherence conditions of a monoidal product, namely, that the following diagrams commute for all $x, y, z, w \in \mathcal{C}_1$:

$$\begin{tikzcd} & x \otimes y & \\ x \otimes (1 \otimes y) \arrow[ur, "x \otimes l_y"] \arrow[rr, "a_{x,1,y}"'] & & (x \otimes 1) \otimes y \arrow[ul, "r_y \otimes y"'] \end{tikzcd} \tag{3.26}$$

$$\begin{tikzcd} x \otimes (y \otimes (z \otimes w)) \arrow[rr, "a_{x,y,z \otimes w}"] \arrow[d, "x \otimes a_{y,z,w}"'] & & (x \otimes y) \otimes (z \otimes w) \arrow[d, "a_{x \otimes y,z,w}"] \\ x \otimes ((y \otimes z) \otimes w) \arrow[dr, "a_{x,y \otimes z,w}"'] & & ((x \otimes y) \otimes z) \otimes w \\ & (x \otimes (y \otimes z)) \otimes w\ . \arrow[ur, "a_{x,y,z} \otimes w"'] & \end{tikzcd} \tag{3.27}$$

First, we will show that (3.26) commutes. By the definitions of a, l, and r, together with (3.23) and (3.24), we can decompose the diagram into the outermost diagram below:

$$\begin{tikzcd} & \Phi\mu(x, \Phi y) \arrow[r, "\mathcal{R}_{x,y}"] & \Phi\mu(x, y) & \Phi\mu(\Phi x, y) \arrow[l, "\mathcal{L}_{x,y}"'] & \\ \Phi\mu(x, \Phi\mu(1, y)) \arrow[ur, "\Phi\mu(x, \Phi\lambda_y)"] \arrow[r, "\mathcal{R}_{x,\mu(1,y)}"'] & \Phi\mu(x, \mu(1, y)) \arrow[ur, "\Phi\mu(x,\lambda_y)"'] \arrow[rr, "\Phi\alpha_{x,1,y}"'] & & \Phi\mu(\mu(x, 1), y) \arrow[ul, "\Phi\mu(\rho_x,y)"] & \Phi\mu(\Phi\mu(x, 1), y) \arrow[l, "\mathcal{L}_{\mu(x,1),y}"] \arrow[ul, "\Phi\mu(\Phi\rho_x,y)"'] \\ \Phi\mu(x, \Phi\mu(I, y)) \arrow[u, "\Phi\mu(x,\mathcal{L}_{1,y})"] \arrow[r, "\mathcal{R}_{x,\mu(I,y)}"'] & \Phi\mu(x, \mu(I, y)) \arrow[rr, "\Phi\alpha_{x,I,y}"'] & & \Phi\mu(\mu(x, I), y) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,I),y}"'] & \Phi\mu(\Phi\mu(x, I), y)\ . \arrow[u, "\Phi\mu(\mathcal{R}_{x,1},y)"'] \end{tikzcd}$$

But each inner subdiagram above commutes: the top left and right quadrilaterals commute by the naturality of $\mathcal{L}$ and $\mathcal{R}$; the top middle triangle is precisely the image of (3.3) under Φ; and the bottom rectangle is (3.22). Therefore the entire diagram commutes.

It remains to be shown that (3.27) commutes. Again, we decompose the diagram into subdiagrams to form the (somewhat menacing) one below:

$$\begin{tikzcd}[column sep=small] \Phi\mu(x, \Phi\mu(y, \Phi\mu(z, w))) \arrow[r, "\mathcal{R}_{x,\mu(y,\Phi\mu(z,w))}"] \arrow[d, "\Phi\mu(x,\mathcal{R}_{y,\mu(z,w)})"'] & \Phi\mu(x, \mu(y, \Phi\mu(z, w))) \arrow[r, "\Phi\alpha_{x,y,\Phi\mu(z,w)}"] & \Phi\mu(\mu(x, y), \Phi\mu(z, w)) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,y),\Phi\mu(z,w)}"] \arrow[d, "\mathcal{R}_{\mu(x,y),\mu(z,w)}"'] & \Phi\mu(\Phi\mu(x, y), \Phi\mu(z, w)) \arrow[d, "\mathcal{R}_{\Phi\mu(x,y),\mu(z,w)}"] & \\ \Phi\mu(x, \Phi\mu(y, \mu(z, w))) \arrow[r, "\mathcal{R}_{x,\mu(y,\mu(z,w))}"'] \arrow[dd, "\Phi\mu(x,\Phi\alpha_{y,z,w})"'] & \Phi\mu(x, \mu(y, \mu(z, w))) \arrow[r, "\Phi\alpha_{x,y,\mu(z,w)}"'] \arrow[dd, "\Phi\mu(x,\alpha_{y,z,w})"'] & \Phi\mu(\mu(x, y), \mu(z, w)) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,y),\mu(z,w)}"'] \arrow[d, "\Phi\alpha_{\mu(x,y),z,w}"'] & \Phi\mu(\Phi\mu(x, y), \mu(z, w)) \arrow[r, "\Phi\alpha_{\Phi\mu(x,y),z,w}"'] & \Phi\mu(\mu(\Phi\mu(x, y), z), w) \arrow[d, "\mathcal{L}^{-1}_{\mu(\Phi\mu(x,y),z),w}"] \\ & & \Phi\mu(\mu(\mu(x, y), z), w) \arrow[r, "\mathcal{L}^{-1}_{\mu(\mu(x,y),z),w}"'] & \Phi\mu(\Phi\mu(\mu(x, y), z), w) \arrow[r, "\Phi\mu(\mathcal{L}^{-1}_{\mu(x,y),z},w)"'] & \Phi\mu(\Phi\mu(\Phi\mu(x, y), z), w) \\ \Phi\mu(x, \Phi\mu(\mu(y, z), w)) \arrow[r, "\mathcal{R}_{x,\mu(\mu(y,z),w)}"'] \arrow[d, "\Phi\mu(x,\mathcal{L}^{-1}_{\mu(y,z),w})"'] & \Phi\mu(x, \mu(\mu(y, z), w)) \arrow[r, "\Phi\alpha_{x,\mu(y,z),w}"'] & \Phi\mu(\mu(x, \mu(y, z)), w) \arrow[u, "\Phi\mu(\alpha_{x,y,z},w)"] \arrow[r, "\mathcal{L}^{-1}_{\mu(x,\mu(y,z)),w}"'] & \Phi\mu(x, \mu(\Phi\mu(y, z), w)) \arrow[r, "\Phi\mu(\Phi\alpha_{x,y,z},w)"'] & \Phi\mu(\Phi\mu(\mu(x, y), z), w) \arrow[u, "\Phi\mu(\mathcal{L}^{-1}_{\mu(x,y),z},w)"'] \\ \Phi\mu(x, \Phi\mu(\Phi\mu(y, z), w)) \arrow[r, "\mathcal{R}_{x,\mu(\Phi\mu(y,z),w)}"'] & \Phi\mu(x, \mu(\Phi\mu(y, z), w)) \arrow[r, "\Phi\alpha_{x,\Phi\mu(y,z),w}"'] & \Phi\mu(\mu(x, \Phi\mu(y, z)), w) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,\Phi\mu(y,z)),w}"'] & \Phi\mu(\Phi\mu(x, \Phi\mu(y, z)), w)\ . \arrow[u, "\Phi\mu(\mathcal{R}_{x,\mu(y,z)},w)"'] & \end{tikzcd}$$

Once again, we will find that each subdiagram commutes. The top-left hexagon is just (3.20), the two middle-right hexagons are (3.21), and the top-right square is (3.19). Additionally, the middle-left square commutes by the naturality of $\mathcal{R}$. Finally, the central pentagon is the image of (3.4) under Φ, and thus also commutes. Therefore the entire diagram commutes, completing the proof. □

Since we have generalized the notions of symmetric and closed monoidal products to the graded setting, it would be impolite not to do the same for internalizations:

Definition 3.28. Let $(\mathcal{C}_\bullet, \mu, 1)$ be a symmetric graded monoidal category with braiding σ, equipped with an internalization $(\Phi, \mathcal{L}, \mathcal{R})$. Then Φ is a *symmetric internalization* if, for all $x, y \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
\Phi\mu(\Phi x, y) \arrow[r, "\Phi\sigma_{\Phi x,y}"] \arrow[d, "\mathcal{L}_{x,y}"'] & \Phi\mu(y, \Phi x) \arrow[d, "\mathcal{R}_{y,x}"] \\
\Phi\mu(x, y) \arrow[r, "\Phi\sigma_{x,y}"'] & \Phi\mu(y, x)
\end{tikzcd}$$

commutes.

Corollary 3.29. Let $(\mathcal{C}_\bullet, \mu, 1)$ be a symmetric graded monoidal category with braiding σ, equipped with a symmetric internalization $(\Phi, \mathcal{L}, \mathcal{R})$. Then the natural isomorphism

$$s_{x,y} := \Phi_2(\sigma_{1,1})_{x,y}$$

defines a braiding on the internalization $\otimes$ of μ with respect to Φ, making $(\mathcal{C}_1, \otimes, I)$ into a symmetric monoidal category.

Proof. We must verify that the natural isomorphism s satisfies the three coherence conditions for a symmetric monoidal category, namely, that the following diagrams commute for all $x, y, z \in \mathcal{C}_1$:

$$\begin{tikzcd}
& x & \\
I \otimes x \arrow[ur, "l_x"] \arrow[rr, "s_{1,x}"'] & & x \otimes I \arrow[ul, "r_x"']
\end{tikzcd} \tag{3.30}$$

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "x \otimes s_{y,z}"] \arrow[d, "a_{x,y,z}"'] & x \otimes (z \otimes y) \arrow[d, "a_{x,z,y}"] \\
(x \otimes y) \otimes z \arrow[d, "s_{x \otimes y, z}"'] & (x \otimes z) \otimes y \arrow[d, "s_{x,z} \otimes y"] \\
z \otimes (x \otimes y) \arrow[r, "a_{z,x,y}"'] & (z \otimes x) \otimes y
\end{tikzcd} \tag{3.31}$$

$$\begin{tikzcd}
x \otimes y \arrow[rr, equal] \arrow[dr, "s_{x,y}"'] & & x \otimes y \\
& y \otimes x, \arrow[ur, "s_{y,x}"'] &
\end{tikzcd} \tag{3.32}$$

where a, l, and r are defined in Theorem 3.25. First, we observe that (3.30) commutes by the commutativity of

$$\begin{tikzcd}
 & \Phi x & \\
\Phi\mu(1,x) \arrow[ur, "\Phi\lambda_x"] \arrow[rr, "\Phi\sigma_{1,x}"] & & \Phi\mu(x,1) \arrow[ul, "\Phi\rho_x"'] \\
\Phi\mu(\Phi(1),x) \arrow[u, "\mathcal{L}_{1,x}"] \arrow[rr, "\Phi\sigma_{\Phi(1),x}"'] & & \Phi\mu(x,\Phi(1)) \arrow[u, "\mathcal{R}_{x,1}"'] .
\end{tikzcd}$$

Each subdiagram commutes – the triangle is image of (3.11) under Φ, and the square commutes by Definition 3.28 – and, post-composition with ι_x yields (3.30).

Next, we will show that (3.31) commutes. As usual, we begin by decomposing the diagram into smaller subdiagrams:

$$\begin{tikzcd}
 & \Phi\mu(x,\Phi\mu(y,z)) \arrow[r, "\Phi\mu(x,\Phi\sigma_{y,z})"] \arrow[d, "\mathcal{R}_{x,\mu(y,z)}"'] & \Phi\mu(x,\Phi\mu(z,y)) \arrow[d, "\mathcal{R}_{x,\mu(z,y)}"] & \\
 & \Phi\mu(x,\mu(y,z)) \arrow[r, "\Phi\mu(x,\sigma_{y,z})"'] \arrow[d, "\Phi\alpha_{x,y,z}"'] & \Phi\mu(x,\mu(z,y)) \arrow[d, "\Phi\alpha_{x,z,y}"] & \\
\Phi\mu(\Phi\mu(x,y),z) \arrow[d, "\Phi\sigma_{\Phi\mu(x,y),z}"'] & \Phi\mu(\mu(x,y),z) \arrow[l, "\mathcal{L}^{-1}_{\mu(x,y),z}"'] \arrow[d, "\Phi\sigma_{\mu(x,y),z}"'] & \Phi\mu(\mu(x,z),y) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,z),y}"] \arrow[d, "\Phi\mu(\sigma_{x,z},y)"] & \Phi\mu(\Phi\mu(x,z),y) \arrow[d, "\Phi\mu(\Phi\sigma_{x,z},y)"] \\
\Phi\mu(z,\Phi\mu(x,y)) \arrow[r, "\mathcal{R}_{z,\mu(x,y)}"'] & \Phi\mu(z,\mu(x,y)) \arrow[r, "\Phi\alpha_{z,x,y}"'] & \Phi\mu(\mu(z,x),y) \arrow[r, "\mathcal{L}^{-1}_{\mu(z,x),y}"'] & \Phi\mu(\Phi\mu(z,x),y) .
\end{tikzcd}$$

The topmost square commutes by the naturality of $\mathcal{R}$. The bottom-left square commutes by Definition 3.28. The bottom-right square commutes by the naturality of $\mathcal{L}$. Finally, the middle hexagon commutes since it is the image of (3.12) under Φ. Thus, (3.31) commutes.

Finally, it is immediate that (3.32) commutes, since it is just the image of (3.13) under Φ. Therefore s defines a braiding for $\otimes$ which makes $(\mathcal{C}_1, \otimes, I)$ into a symmetric monoidal category. □

Definition 3.33. Let $(\mathcal{C}_\bullet, \mu, 1)$ be a closed graded monoidal category with internal hom $\mathscr{F}$, equipped with an internalization $(\Phi, \mathcal{L}, \mathcal{R})$. Then Φ is a *closed internalization* if, for each $m \in \mathbb{N}$, there exists a right adjoint $\Psi_m : \mathcal{C}_1 \to \mathcal{C}_m$ to Φ_m. For brevity, we write $\Phi \dashv \Psi$ to denote the adjunction.

Corollary 3.34. Let $(\mathcal{C}_\bullet, \mu, 1)$ be a closed graded monoidal category with internal hom $\mathscr{F}$, equipped with a closed internalization $(\Phi, \mathcal{L}, \mathcal{R})$, with $\Phi \dashv \Psi$. Then $(\mathcal{C}_1, \otimes, I)$ is a closed monoidal category with internal hom

$$F(x,y) := \mathscr{F}_{1,1}(x, \Psi_2 y)$$

for each $x, y \in \mathcal{C}_1$.

Proof. Let $x, y, z \in \mathcal{C}_1$. Then the claim follows immediately by composing the natural isomorphisms

$$\begin{aligned}
\mathcal{C}_1(x \otimes y, z) &= \mathcal{C}_1(\Phi_2 \mu_{1,1}(x, y), z) \\
&\cong \mathcal{C}_2(\mu_{1,1}(x, y), \Psi_2 z) \\
&\cong \mathcal{C}_1(x, \mathscr{F}_{1,1}(y, \Psi_2 z)) \\
&= \mathcal{C}_1(x, F(y, z)).
\end{aligned}$$

Diagramatically, this is expressed by the pasting diagram below (where, as noted in Remark 3.16, one can replace **Set** with the appropriate base category in the enriched setting):

$$\begin{tikzcd}
 & \mathcal{C}_2^{\mathrm{op}} \times \mathcal{C}_1 \arrow[r, "\Phi_2 \times \mathcal{C}_1"] \arrow[dr, "\mathcal{C}_2^{\mathrm{op}} \times \psi_2"'] & \mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_1 \arrow[d, phantom, "\Downarrow\wr"] \arrow[dr, "\mathcal{C}_1(-,-)"] & \\
\mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_1 \arrow[ur, "\mu_{1,1} \times \mathcal{C}_1"] \arrow[dr, "\mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_1^{\mathrm{op}} \times \psi_2"'] & \| & \mathcal{C}_2^{\mathrm{op}} \times \mathcal{C}_2 \arrow[r, "\mathcal{C}_2(-,-)"] \arrow[d, phantom, "\Downarrow\wr"] & \mathbf{Set} \\
 & \mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_2 \arrow[r, "\mathcal{C}_1^{\mathrm{op}} \times \mathscr{F}_{1,1}"'] \arrow[ur, "\mu_{1,1} \times \mathcal{C}_2"] & \mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_1 \arrow[ur, "\mathcal{C}_1(-,-)"'] &
\end{tikzcd}.$$

□

In Section 5, we'll use the results of this section to recover two constructions of point-set smash products of spectra, Day convolution and the smash product of S-modules, and show that these are closed and symmetric purely via the formal machinery of internalization.

4. Categorical Properties of Internalization

In this section, we outline the basic (higher) categorical properties of graded monoidal categories and internalized categories. With suitable notions of functors and natural transformations, we will see that these form 2-categories equipped with compatible adjunctions with classical structures. In particular, internalization transforms a graded monoidal category *functorially* into an ordinary monoidal category (Proposition 4.20), and every monoidal category can be lifted to a *free* internalized category whose internalization recovers the original monoidal product in degree one (Theorem 4.21, Theorem 4.23).

4.1. **The 2-Category of Graded Monoidal Categories.**
We begin by defining (lax, strong, and strict) graded monoidal functors, which is directly analogous to the corresponding definition(s) for ordinary monoidal categories [6, Chapter XI].

Definition 4.1. A *lax graded monoidal functor* $f_\bullet : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ consists of

(1) a functor $f_m : \mathcal{C}_m \to \mathcal{D}_m$ for each $m \in \mathbb{N}$,
(2) a natural transformation

$$\zeta^{m,n} : \mu^{\mathcal{D}}_{m,n}(f_m, f_n) \Rightarrow f_{m+n} \mu^{\mathcal{C}}_{m,n}$$

for each $m, n \in \mathbb{N}$,
(3) a morphism $\nu : 1^{\mathcal{D}} \to f_0(1^{\mathcal{C}})$,

such that the following diagrams commute (for all $x \in \mathcal{C}_m, y \in \mathcal{C}_n$, and $z \in \mathcal{C}_r$):

$$\begin{tikzcd}
\mu^{\mathcal{D}}(f_m(x), \mu^{\mathcal{D}}(f_n(y), f_r(z))) \arrow[r, "\alpha^{\mathcal{D}}"] \arrow[d, "\mu^{\mathcal{D}}(\mathrm{id}, \zeta^{n,r})"'] & \mu^{\mathcal{D}}(\mu^{\mathcal{D}}(f_m(x), f_n(y)), f_r(z)) \arrow[d, "\mu^{\mathcal{D}}(\zeta^{m,n}, \mathrm{id})"] \\
\mu^{\mathcal{D}}(f_m(x), f_{n+r}\mu^{\mathcal{C}}(y,z)) \arrow[d, "\zeta^{m,n+r}"'] & \mu^{\mathcal{D}}(f_{m+n}\mu^{\mathcal{C}}(x,y), f_r(z)) \arrow[d, "\zeta^{m+n,r}"] \\
f_{m+n+r}\mu^{\mathcal{C}}(x, \mu^{\mathcal{C}}(y,z)) \arrow[r, "f_{m+n+r}\alpha^{\mathcal{C}}"'] & f_{m+n+r}\mu^{\mathcal{C}}(\mu^{\mathcal{C}}(x,y), z)
\end{tikzcd}$$

$$\begin{tikzcd}
\mu^{\mathcal{D}}(1^{\mathcal{D}}, f_m(x)) \arrow[r, "\mu^{\mathcal{D}}(\nu, \mathrm{id})"] \arrow[d, "\lambda^{\mathcal{D}}"'] & \mu^{\mathcal{D}}(f_0(1^{\mathcal{C}}), f_m(x)) \arrow[d, "\zeta^{0,m}"] \\
f_m(x) & f_m\mu^{\mathcal{C}}(1^{\mathcal{C}}, x) \arrow[l, "f_m\lambda^{\mathcal{C}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mu^{\mathcal{D}}(f_m(x), 1^{\mathcal{D}}) \arrow[r, "\mu^{\mathcal{D}}(\mathrm{id}, \nu)"] \arrow[d, "\rho^{\mathcal{D}}"'] & \mu^{\mathcal{D}}(f_m(x), f_0(1^{\mathcal{C}})) \arrow[d, "\zeta^{m,0}"] \\
f_m(x) & f_m\mu^{\mathcal{C}}(x, 1^{\mathcal{C}}) \arrow[l, "f_m\rho^{\mathcal{C}}"]
\end{tikzcd} .$$

If ν and each $\zeta^{m,n}$ are isomorphisms then $f_\bullet$ is a *strong graded monoidal functor*, and if they are identity morphisms then $f_\bullet$ is a *strict graded monoidal functor*.

The identity $\mathrm{id}_\bullet : \mathcal{C}_\bullet \to \mathcal{C}_\bullet$ is defined to be the componentwise identity in each degree, and is a strict graded monoidal functor. The composite $(gf)_\bullet$ of a pair of lax (resp. strong, strict) graded monoidal functors $f_\bullet : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$, $g_\bullet : \mathcal{D}_\bullet \to \mathcal{E}_\bullet$ is obtained by componentwise composition over the grading, while the composites

$$\mu^{\mathcal{E}}_{m,n}(g_m f_m(x), g_n f_n(y)) \to g_{m+n}(\mu^{\mathcal{D}}_{m,n}(f_m(x), f_n(y))) \to g_{m+n} f_{m+n} \mu^{\mathcal{C}}_{m,n}(x,y)$$

and

$$1^{\mathcal{E}} \to g_0(1^{\mathcal{D}}) \to g_0 f_0(1^{\mathcal{C}})$$

define the structure maps making $(gf)_\bullet$ a lax (resp. strong, strict) functor. We elide the verification of the coherence diagrams; the argument is identical to the classical case, except extra care must be taken to track the degrees of each object.

Remark 4.2. While the choice to favor any one strength of graded monoidal functor is a matter of taste, for our purposes it will be largely irrelevant. We will henceforth assume all (graded) monoidal functors are lax, but note that throughout this section they could be made strong or strict without changing any of the subsequent results. In light of this remark, and for brevity, we will simply refer to these as "graded monoidal functors."

Definition 4.3. Graded monoidal categories[6] and graded monoidal functors between them form a category, which we denote **GrMonCat**.

As is typical for categories whose morphisms are some flavor of functor, a more complete description of **GrMonCat** is as a 2-category [6, Section XII.3]. To make sense of this, we first need a notion of natural transformations between graded monoidal functors, which, predictably, is just the graded version of monoidal transformations [6, Chapter XI]:

[6]As usual when referring to categories of categories, we must fix an upper bound on the size of the categories in question to avoid size issues. To be precise, **GrMonCat** is the category of *small* graded monoidal categories (wherein each component is a small category).

Definition 4.4. Given graded monoidal functors $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$, a *graded monoidal (natural) transformation* $\beta_\bullet : f_\bullet \Rightarrow g_\bullet$ consists of a natural transformation $\beta_m : f_m \Rightarrow g_m$ for each $m \in \mathbb{N}$ such that the following diagrams commute (for all $x \in \mathcal{C}_m, y \in \mathcal{C}_n$):

$$\begin{tikzcd} & 1^{\mathcal{D}} \arrow[dl, "\nu"'] \arrow[dr, "\nu"] & \\ f_0(1^{\mathcal{C}}) \arrow[rr, "\beta_0"'] & & g_0(1^{\mathcal{C}}) \end{tikzcd}$$

$$\begin{tikzcd} \mu^{\mathcal{D}}(f_m(x), f_n(y)) \arrow[r, "\mu^{\mathcal{D}}(\beta_m,\beta_n)"] \arrow[d, "\zeta^{m,n}"'] & \mu^{\mathcal{D}}(g_m(x), g_n(y)) \arrow[d, "\zeta^{m,n}"] \\ f_{m+n}\mu^{\mathcal{C}}(x,y) \arrow[r, "\beta_{m+n}"'] & g_{m+n}\mu^{\mathcal{C}}(x,y) \end{tikzcd}$$

(We abuse notation slightly by denoting the structure maps of both $f_\bullet$ and $g_\bullet$ using the same symbols ν and ζ, in an admittedly futile attempt at minimizing clutter. Throughout this section, when it is clear which structure maps are being referenced, we will renew this abuse.)

With graded monoidal functors as 1-morphisms and graded monoidal natural transformations as 2-morphisms, **GrMonCat** forms a 2-category. Since graded monoidal functors and natural transformations are defined componentwise, it is clear that isolating any single degree m defines a 2-functor **GrMonCat** $\to$ **Cat**, where **Cat** denotes the 2-category of (small) categories. We give special attention to the case when $m = 1$:

Proposition 4.5. There is a 2-functor $\mathrm{Forget}_{\mathrm{Gr}}$: **GrMonCat** $\to$ **Cat** which sends a graded monoidal category $(\mathcal{C}_\bullet, \mu, 1)$ to its degree one component $\mathcal{C}_1$.

Much like one can construct a free monoidal category from any category via the finite list monad [6, Section VII.3, Theorem 2], there is an analogous *free graded monoidal product* on any category, where the grading is given by list length. As one might expect, this construction is 2-functorial:

Proposition 4.6. There is a 2-functor $\mathrm{Free}_{\mathrm{Gr}}$: **Cat** $\to$ **GrMonCat** which carries a category $\mathcal{C}$ to the graded monoidal category $\mathrm{Free}_{\mathrm{Gr}}(\mathcal{C})_\bullet$ defined by

$$\mathrm{Free}_{\mathrm{Gr}}(\mathcal{C})_m := \mathcal{C}^m$$

equipped with the empty string $1^{\mathrm{Free}} \in \mathcal{C}^0$ as the unit 1^{Free} and graded monoidal product μ^{Free} given by concatenation:

$$\mu^{\mathrm{Free}}_{m,n}((x_1, \dots, x_m), (y_1, \dots, y_n)) := (x_1, \dots, x_m, y_1, \dots, y_n).$$

Proof. The structure maps for the graded monoidal product μ^{Free} are trivial (that is, $\mathrm{Free}_{\mathrm{Gr}}(\mathcal{C})_\bullet$ is a *strict* graded monoidal product), and thus the axioms are free.

For a functor $f : \mathcal{C} \to \mathcal{D}$ we define

$$\mathrm{Free}_{\mathrm{Gr}}(f)_m := f^m : \mathcal{C}^m \to \mathcal{D}^m$$

that is, by termwise application of f, which is clearly compatible with the concatenation product and empty string unit of $\mathrm{Free}_{\mathrm{Gr}}(\mathcal{C})_\bullet$.

Similarly, for a natural transformation $\beta : f \Rightarrow g$, let $\mathrm{Free}_{\mathrm{Gr}}(\beta)_m := \beta^m$, which one easily checks defines a graded monoidal transformation $\mathrm{Free}_{\mathrm{Gr}}(f)_\bullet \Rightarrow \mathrm{Free}_{\mathrm{Gr}}(g)_\bullet$. Clearly Free is 2-functorial. $\square$

This highlights the significance of the degree one component – lists of length one contain exactly the data of the original category, which is why we view $\mathrm{Forget}_{\mathrm{Gr}}$ as a kind of "forgetful" functor. In other words:

Proposition 4.7. $\mathrm{Free}_{\mathrm{Gr}}$ a section of $\mathrm{Forget}_{\mathrm{Gr}}$.

Proof. From the definitions in Propositions 4.5 and 4.6 we verify directly that

$$(\mathrm{Forget}_{\mathrm{Gr}} \circ \mathrm{Free}_{\mathrm{Gr}})(\mathcal{C}) = \mathrm{Free}_{\mathrm{Gr}}(\mathcal{C})_1 = \mathcal{C}^1 = \mathcal{C}.$$

on objects $\mathcal{C} \in \mathbf{Cat}$. The cases for 1-morphisms (functors) and 2-morphisms (natural transformations) are identical. □

The free graded monoidal category above is indeed "free," in the sense that it is left adjoint to $\mathrm{Forget}_{\mathrm{Gr}}$.

Proposition 4.8. There is a (strict) 2-adjunction of 2-functors

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, bend left, "\mathrm{Free}_{\mathrm{Gr}}"] \arrow[r, phantom, "\perp"] & \mathbf{GrMonCat} \arrow[l, bend left, "\mathrm{Forget}_{\mathrm{Gr}}"] \end{tikzcd} .$$

Proof. To establish a 2-adjunction $\mathrm{Free}_{\mathrm{Gr}} \dashv \mathrm{Forget}_{\mathrm{Gr}}$, we construct the unit η and counit ϵ as follows: in light of Proposition 4.7, we define the component $\eta_{\mathcal{C}}$ to simply be the identity functor on $\mathcal{C}$. Given $(\mathcal{D}_\bullet, \mu^{\mathcal{D}}, 1^{\mathcal{D}}) \in \mathbf{GrMonCat}$, we define the m^{th} graded component of the counit $(\epsilon_{\mathcal{D}_\bullet})_m : \mathcal{D}_1^m \to \mathcal{D}_m$ by

$$(\epsilon_{\mathcal{D}_\bullet})_m(x_1, \ldots, x_m) := \mu^{\mathcal{D}}(x_1, \ldots, x_m)$$

(where we fix a bias for the m-fold iterated $\mu^{\mathcal{D}}$ product, say by associating to the right) if $m > 1$ and take

$$\begin{aligned} (\epsilon_{\mathcal{D}_\bullet})_1(x) &:= x \\ (\epsilon_{\mathcal{D}_\bullet})_0(*) &:= 1^{\mathcal{D}}. \end{aligned}$$

Then for $(x_1, \ldots, x_m) \in \mathcal{C}^m = \mathrm{Free}_{\mathrm{Gr}}(\mathcal{C})_m$ we have

$$\begin{aligned} (\epsilon_{\mathrm{Free}_{\mathrm{Gr}}(\mathcal{C})_\bullet} \circ \mathrm{Free}_{\mathrm{Gr}}(\eta_{\mathcal{C}}))_m(x_1, \ldots, x_m) &= \epsilon_{\mathrm{Free}_{\mathrm{Gr}}(\mathcal{C})_\bullet}(\eta_{\mathcal{C}}(x_1), \ldots, \eta_{\mathcal{C}}(x_m)) \\ &= \mu^{\mathrm{Free}}(x_1, \ldots, x_m) \\ &= (x_1, \ldots, x_m), \end{aligned}$$

so

$$\epsilon_{\mathrm{Free}_{\mathrm{Gr}}(\mathcal{C})_\bullet} \circ \mathrm{Free}_{\mathrm{Gr}}(\eta_{\mathcal{C}})_\bullet = \mathrm{id}_{\mathrm{Free}_{\mathrm{Gr}}(\mathcal{C})_\bullet}.$$

On the other hand, for $y \in \mathcal{D}_1 = \mathrm{Forget}_{\mathrm{Gr}}(\mathcal{D}_\bullet)$ we have

$$(\mathrm{Forget}_{\mathrm{Gr}}(\epsilon_{\mathcal{D}_\bullet}) \circ \eta_{\mathrm{Forget}_{\mathrm{Gr}}(\mathcal{D}_\bullet)})(y) = \mathrm{Forget}_{\mathrm{Gr}}(\epsilon_{\mathcal{D}_\bullet})(y) = (\epsilon_{\mathcal{D}_\bullet})_1(y) = y.$$

An identical argument verifies the equations for morphisms, so indeed η and ϵ define a (strict) 2-adjunction $\mathrm{Free}_{\mathrm{Gr}} \dashv \mathrm{Forget}_{\mathrm{Gr}}$. □

Corollary 4.9. $\mathrm{Free}_{\mathrm{Gr}}$ embeds $\mathbf{Cat}$ as a coreflective subcategory of $\mathbf{GrMonCat}$.

Proof. Clearly $\mathrm{Free}_{\mathrm{Gr}}$ is injective on objects, since we can recover the category $\mathcal{C}$ from the degree one component of $\mathrm{Free}_{\mathrm{Gr}}(\mathcal{C})_\bullet$; the same is clearly true for functors and natural transformations. Thus $\mathrm{Free}_{\mathrm{Gr}}$ is an embedding of 2-categories. The unit of the adjunction in Proposition 4.8 is the identity (so in particular is an isomorphism), making this embedding coreflective. $\square$

4.2. **The 2-Category of Internalized Categories.** Having established the relationship between the 2-categories **Cat** and **GrMonCat**, we now turn our attention to *internalized* graded monoidal categories. As one might hope, these too assemble into a 2-category **IntMonCat**. This requires an appropriate extension of graded monoidal functors and natural transformations to accomodate the additional structure of an internalization.

Definition 4.10. Let $\mathcal{C}_\bullet$ and $\mathcal{D}_\bullet$ be graded monoidal categories equipped with internalizations $\Phi^{\mathcal{C}}$ and $\Phi^{\mathcal{D}}$, respectively. A *lax internalized (monoidal) functor* $f_\bullet : (\mathcal{C}_\bullet, \Phi^{\mathcal{C}}) \to (\mathcal{D}_\bullet, \Phi^{\mathcal{D}})$ is a lax graded monoidal functor $\mathcal{C}_\bullet \to \mathcal{D}_\bullet$ together with natural transformations

$$\xi^m : \Phi^{\mathcal{D}}_m f_m \Rightarrow f_1 \Phi^{\mathcal{C}}_m$$

such that the following diagrams commute (for all $x \in \mathcal{C}_m, y \in \mathcal{C}_n$, and $z \in \mathcal{C}_r$):

$$\begin{array}{ccc}
\Phi^{\mathcal{D}}_{n+1}\mu^{\mathcal{D}}(\Phi^{\mathcal{D}}_m f_m(x), f_n(y)) & \xrightarrow{\Phi^{\mathcal{D}}_{n+1}\mu^{\mathcal{D}}(\xi^m,\mathrm{id})} & \Phi^{\mathcal{D}}_{n+1}\mu^{\mathcal{D}}(f_1\Phi^{\mathcal{C}}_m(x), f_n(y)) \\
\downarrow{\scriptstyle \mathcal{L}^{\mathcal{D}}} & & \downarrow{\scriptstyle \Phi^{\mathcal{D}}_{n+1}\zeta^{1,n}} \\
\Phi^{\mathcal{D}}_{m+n}\mu^{\mathcal{D}}(f_m(x), f_n(y)) & & \Phi^{\mathcal{D}}_{n+1} f_{n+1}\mu^{\mathcal{C}}(\Phi^{\mathcal{C}}_m(x), y) \\
\downarrow{\scriptstyle \Phi^{\mathcal{D}}_{m+n}\zeta^{m,n}} & & \downarrow{\scriptstyle \xi^{n+1}} \\
\Phi^{\mathcal{D}}_{m+n} f_{m+n}\mu^{\mathcal{C}}(x,y) & & f_1\Phi^{\mathcal{C}}_{n+1}\mu^{\mathcal{C}}(\Phi^{\mathcal{C}}_m(x), y) \\
& \searrow{\scriptstyle \xi^{m+n}} \quad \swarrow{\scriptstyle f_1\mathcal{L}^{\mathcal{C}}} & \\
& f_1\Phi^{\mathcal{C}}_{m+n}\mu^{\mathcal{C}}(x,y) &
\end{array} \tag{4.11}$$

$$\begin{array}{ccc}
\Phi^{\mathcal{D}}_{m+1}\mu^{\mathcal{D}}(f_m(x), \Phi^{\mathcal{D}}_n f_n(y)) & \xrightarrow{\Phi^{\mathcal{D}}_{m+1}\mu^{\mathcal{D}}(\mathrm{id},\xi^n)} & \Phi^{\mathcal{D}}_{m+1}\mu^{\mathcal{D}}(f_m(x), f_1\Phi^{\mathcal{C}}_n(y)) \\
\downarrow{\scriptstyle \mathcal{R}^{\mathcal{D}}} & & \downarrow{\scriptstyle \Phi^{\mathcal{D}}_{m+1}\zeta^{m,1}} \\
\Phi^{\mathcal{D}}_{m+n}\mu^{\mathcal{D}}(f_m(x), f_n(y)) & & \Phi^{\mathcal{D}}_{n+1} f_{n+1}\mu^{\mathcal{C}}(\Phi^{\mathcal{C}}_m(x), y) \\
\downarrow{\scriptstyle \Phi^{\mathcal{D}}_{m+n}\zeta^{m,n}} & & \downarrow{\scriptstyle \xi^{n+1}} \\
\Phi^{\mathcal{D}}_{m+n} f_{m+n}\mu^{\mathcal{C}}(x,y) & & f_1\Phi^{\mathcal{C}}_{m+1}\mu^{\mathcal{C}}(x, \Phi^{\mathcal{C}}_n(y)) \\
& \searrow{\scriptstyle \xi^{m+n}} \quad \swarrow{\scriptstyle f_1\mathcal{R}^{\mathcal{C}}} & \\
& f_1\Phi^{\mathcal{C}}_{m+n}\mu^{\mathcal{C}}(x,y) &
\end{array} \tag{4.12}$$

$$\begin{array}{ccc}
& \Phi^{\mathcal{D}}_1 f_1 \Phi^{\mathcal{C}}_1(x) & \\
\swarrow{\scriptstyle \Phi^{\mathcal{D}}_1 f_1 \iota^{\mathcal{C}}} & & \searrow{\scriptstyle \iota^{\mathcal{D}}} \\
\Phi^{\mathcal{D}}_1 f_1(x) & \xrightarrow[\xi^1]{} & f_1(x)
\end{array} \tag{4.13}$$

If $f_\bullet$ is a strong graded monoidal functor and each natural transformation ξ^m is an isomorphism we say $f_\bullet$ is a *strong internalized (monoidal) functor*; if $f_\bullet$ is a strict graded monoidal functor and the ξ^m are all identities then $f_\bullet$ is a *strict internalized (monoidal) functor*.

Going forward, we assume the same choice of strength for all internalized functors as graded monoidal functors. As above, we default to the weakest case of all lax functors, but note that if every monoidal functor in sight is strengthened to the strong or strict cases the following results remain valid.

Given an internalized category $\mathcal{C}_\bullet$, we denote by $\mathrm{Int}(\mathcal{C}_\bullet)$ the corresponding monoidal category equipped with the internalized product of Theorem 3.25. The universe is just, and this construction extends to internalized functors:

Proposition 4.14. Let $f_\bullet : (\mathcal{C}_\bullet, \Phi^{\mathcal{C}}) \to (\mathcal{D}_\bullet, \Phi^{\mathcal{D}})$ be an internalized functor. Then, equipping $\mathcal{C}_1$ (resp. $\mathcal{D}_1$) with the internalized monoidal product of Theorem 3.25, the functor $\mathrm{Int}(f) := f_1 : \mathcal{C}_1 \to \mathcal{D}_1$ is a monoidal functor.

Proof. First, we must define structure maps

$$\zeta^{\mathrm{Int}} : \Phi_2^{\mathcal{D}} \mu_{1,1}^{\mathcal{D}}(f, f) \Rightarrow f \Phi_2^{\mathcal{C}} \mu_{1,1}$$
$$\nu^{\mathrm{Int}} : \Phi_0^{\mathcal{D}}(1^{\mathcal{D}}) \to f_1(\Phi_0^{\mathcal{C}}(1^{\mathcal{C}}))$$

for $\mathrm{Int}(f)$. Given $x, y \in \mathcal{C}_1$, we let $\zeta^{\mathrm{Int}}_{x,y}$ be the composite

$$\Phi_2^{\mathcal{D}} \mu_{1,1}^{\mathcal{D}}(f_1(x), f_1(y)) \xrightarrow{\Phi^{\mathcal{D}} \zeta^{1,1}_{x,y}} \Phi_2^{\mathcal{D}} f_2 \mu_{1,1}^{\mathcal{C}}(x, y) \xrightarrow{\xi^2_{\mu^{\mathcal{C}}(x,y)}} f_1 \Phi_2^{\mathcal{C}} \mu_{1,1}^{\mathcal{C}}(x, y),$$

and we take v^{Int} to be the morphism

$$\Phi_0^{\mathcal{D}}(1^{\mathcal{D}}) \xrightarrow{\Phi_0^{\mathcal{D}} \nu} \Phi_0^{\mathcal{D}}(f_0(1^{\mathcal{C}})) \xrightarrow{\xi^0_{1^{\mathcal{C}}}} f_1 \Phi_0^{\mathcal{C}}(1^{\mathcal{C}}).$$

Next, we must check that these structure maps satisfy the coherence conditions of a monoidal functor, namely, that the following diagrams commute (for all $x, y, z \in \mathcal{C}_1$):

$$\begin{tikzcd}
\Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(f(x), \Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(f(y), f(z))) \arrow[r, "a^{\mathcal{D}}"] \arrow[d, "\Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(\mathrm{id}, \zeta^{\mathrm{Int}})"'] & \Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(\Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(f(x), f(y)), f(z)) \arrow[d, "\Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(\zeta^{\mathrm{Int}}, \mathrm{id})"] \\
\Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(f(x), f \Phi_2^{\mathcal{C}} \mu^{\mathcal{C}}(y, z)) \arrow[d, "\zeta^{\mathrm{Int}}"'] & \Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(f \Phi_2^{\mathcal{C}} \mu^{\mathcal{C}}(x, y), f(z)) \arrow[d, "\zeta^{\mathrm{Int}}"] \\
f \Phi_2^{\mathcal{C}} \mu^{\mathcal{C}}(x, \Phi_2^{\mathcal{C}} \mu^{\mathcal{C}}(y, z)) \arrow[r, "f a^{\mathcal{C}}"'] & f \Phi_2^{\mathcal{C}} \mu^{\mathcal{C}}(\Phi_2^{\mathcal{C}} \mu^{\mathcal{C}}(x, y), z)
\end{tikzcd}$$

$$\begin{tikzcd}
\Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(\Phi_0^{\mathcal{D}}(1^{\mathcal{D}}), f(x)) \arrow[r, "\Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(\nu^{\mathrm{Int}}, \mathrm{id})"] \arrow[d, "l^{\mathcal{D}}"'] & \Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(f \Phi_0^{\mathcal{C}}(1^{\mathcal{C}}), f(x)) \arrow[d, "\zeta^{\mathrm{Int}}"] \\
f(x) & f \Phi_2^{\mathcal{C}} \mu^{\mathcal{C}}(\Phi_0^{\mathcal{C}}(1^{\mathcal{C}}), x) \arrow[l, "f l^{\mathcal{C}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi^{\mathcal{D}} \mu^2(f(x), \Phi_0^{\mathcal{D}}(1^{\mathcal{D}})) \arrow[r, "\Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(\mathrm{id}, \nu^{\mathrm{Int}})"] \arrow[d, "r^{\mathcal{D}}"'] & \Phi_2^{\mathcal{D}} \mu^{\mathcal{D}}(f(x), f \Phi_0^{\mathcal{C}}(1^{\mathcal{C}})) \arrow[d, "\zeta^{\mathrm{Int}}"] \\
f(x) & f \Phi_2^{\mathcal{C}} \mu^{\mathcal{C}}(x, \Phi_0^{\mathcal{C}}(1^{\mathcal{C}})) \arrow[l, "f r^{\mathcal{C}}"]
\end{tikzcd}$$

where the (internalized) associator $a^{\mathcal{C}}$ and unitors $l^{\mathcal{C}}, r^{\mathcal{C}}$ for $\mathrm{Int}(\mathcal{C}_\bullet)$, and likewise $a^{\mathcal{D}}, l^{\mathcal{D}}$, and $r^{\mathcal{D}}$ for $\mathrm{Int}(\mathcal{D}_\bullet)$, are as defined in Theorem 3.25.

Unfolding the definitions of each morphism in the hexagonal axiom, verifying commutativity amounts to checking that the perimeter of

$$\begin{tikzcd}
\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(f(x),\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(f(y),f(z))) \arrow[r, "\mathcal{R}^{\mathcal{D}}"] \arrow[d, "\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\mathrm{id},\Phi^{\mathcal{D}}\zeta)"'] & \Phi^{\mathcal{D}}\mu^{\mathcal{D}}(f(x),\mu^{\mathcal{D}}(f(y),f(z))) \arrow[r, "\Phi^{\mathcal{D}}\alpha^{\mathcal{D}}"] \arrow[d, "\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\mathrm{id},\zeta)"'] & \Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\mu^{\mathcal{D}}(f(x),f(y)),f(z)) \arrow[r, "(\mathcal{L}^{\mathcal{D}})^{-1}"] \arrow[d, "\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\zeta,\mathrm{id})"] & \Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(f(x),f(y)),f(z)) \arrow[d, "\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\Phi^{\mathcal{D}}\zeta,\mathrm{id})"] \\
\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(f(x),\Phi^{\mathcal{D}}f\mu^{\mathcal{D}}(f(y),f(z))) \arrow[r, "\mathcal{R}^{\mathcal{D}}"] \arrow[d, "\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\mathrm{id},\xi)"'] & \Phi^{\mathcal{D}}\mu^{\mathcal{D}}(f(x),f\mu^{\mathcal{C}}(y,z)) \arrow[d, "\Phi^{\mathcal{D}}\zeta"'] & \Phi^{\mathcal{D}}\mu^{\mathcal{D}}(f\mu^{\mathcal{C}}(x,y),f(z)) \arrow[r, "(\mathcal{L}^{\mathcal{D}})^{-1}"] \arrow[d, "\Phi^{\mathcal{D}}\zeta"] & \Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\Phi^{\mathcal{D}}f\mu^{\mathcal{C}}(x,y),f(z)) \arrow[d, "\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\xi,\mathrm{id})"] \\
\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(f(x),f\Phi^{\mathcal{C}}\mu^{\mathcal{C}}(y,z)) \arrow[d, "\Phi^{\mathcal{D}}\zeta"'] & \Phi^{\mathcal{D}}f\mu^{\mathcal{C}}(x,\mu^{\mathcal{C}}(y,z)) \arrow[r, "\Phi^{\mathcal{D}}f\alpha^{\mathcal{C}}"'] \arrow[dd, "\xi"'] & \Phi^{\mathcal{D}}f\mu^{\mathcal{C}}(\mu^{\mathcal{C}}(x,y),z) \arrow[dd, "\xi"] & \Phi^{\mathcal{D}}\mu^{\mathcal{D}}(f\Phi^{\mathcal{C}}\mu^{\mathcal{C}}(x,y),f(z)) \arrow[d, "\Phi^{\mathcal{D}}\zeta"] \\
\Phi^{\mathcal{D}}f\mu^{\mathcal{C}}(x,\Phi^{\mathcal{C}}\mu^{\mathcal{C}}(y,z)) \arrow[d, "\xi"'] & & & \Phi^{\mathcal{D}}f\mu^{\mathcal{C}}(\Phi^{\mathcal{C}}\mu^{\mathcal{C}}(x,y),z) \arrow[d, "\xi"] \\
f\Phi^{\mathcal{C}}\mu^{\mathcal{C}}(x,\Phi^{\mathcal{C}}\mu^{\mathcal{C}}(y,z)) \arrow[r, "f\mathcal{R}^{\mathcal{C}}"'] & f\Phi^{\mathcal{C}}\mu^{\mathcal{C}}(x,\mu^{\mathcal{C}}(y,z)) \arrow[r, "f\Phi^{\mathcal{C}}\alpha^{\mathcal{C}}"'] & f\Phi^{\mathcal{C}}\mu^{\mathcal{C}}(\mu^{\mathcal{C}}(x,y),z) \arrow[r, "f(\mathcal{L}^{\mathcal{C}})^{-1}"'] & f\Phi^{\mathcal{C}}\mu^{\mathcal{C}}(\Phi^{\mathcal{C}}\mu^{\mathcal{C}}(x,y),z)
\end{tikzcd}$$

commutes, which follows from the commutativity of each inner sub-diagram:

- the top left square commutes by naturality of $\mathcal{R}^{\mathcal{D}}$;
- the bottom left heptagon is the coherence axiom (4.12) of ξ with the right absorbers;
- the top middle hexagon is the image of the associativity axiom of Definition 4.1 for the graded monoidal functor $f_\bullet$ under $\Phi^{\mathcal{D}}$;
- the bottom middle square commutes by naturality of ξ;
- the top right square commutes by naturality of $(\mathcal{L}^{-1})^{\mathcal{D}}$;
- and the bottom right heptagon is the coherence axiom (4.11) of ξ with the left absorbers (with the absorbers inverted).

Similarly, the compatibility of $(\zeta^{\mathrm{Int}},\nu^{\mathrm{Int}})$ with the left unitors is the outer perimeter of

$$\begin{tikzcd}
\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\Phi^{\mathcal{D}}(1^{\mathcal{D}}),f(x)) \arrow[rrr, "\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\Phi^{\mathcal{D}}\nu,\mathrm{id})"] \arrow[d, "\mathcal{L}^{\mathcal{D}}"'] & & & \Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\Phi^{\mathcal{D}}f(1^{\mathcal{C}}),f(x)) \arrow[r, "\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\xi,\mathrm{id})"] \arrow[d, "\mathcal{L}^{\mathcal{D}}"'] & \Phi^{\mathcal{D}}\mu^{\mathcal{D}}(f\Phi^{\mathcal{C}}(1^{\mathcal{C}}),f(x)) \arrow[dd, "\Phi^{\mathcal{D}}\zeta"] \\
\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(1^{\mathcal{D}},f(x)) \arrow[rrr, "\Phi^{\mathcal{D}}\mu^{\mathcal{D}}(\nu,\mathrm{id})"'] \arrow[d, "\Phi^{\mathcal{D}}\lambda^{\mathcal{D}}"'] & & & \Phi^{\mathcal{D}}\mu^{\mathcal{D}}(f(1^{\mathcal{C}}),f(x)) \arrow[d, "\Phi^{\mathcal{D}}\zeta"'] & \\
\Phi^{\mathcal{D}}f(x) \arrow[d, "\iota^{\mathcal{D}}"'] & & & \Phi^{\mathcal{D}}f\mu^{\mathcal{C}}(1^{\mathcal{C}},x) \arrow[lll, "\Phi^{\mathcal{D}}f\lambda^{\mathcal{C}}"] \arrow[dd, "\xi"'] & \Phi^{\mathcal{D}}f\mu^{\mathcal{C}}(\Phi^{\mathcal{C}}(1^{\mathcal{C}}),x) \arrow[dd, "\xi"] \\
f(x) & \Phi^{\mathcal{D}}f\Phi^{\mathcal{C}}(x) \arrow[ul, "\Phi^{\mathcal{D}}f\iota^{\mathcal{C}}"] \arrow[d, "\iota^{\mathcal{D}}"'] & \Phi^{\mathcal{D}}f\Phi^{\mathcal{C}}\mu^{\mathcal{C}}(1^{\mathcal{C}},x) \arrow[l, "\Phi^{\mathcal{D}}f\Phi^{\mathcal{C}}\lambda^{\mathcal{C}}"] \arrow[ur, "\Phi^{\mathcal{D}}f\iota^{\mathcal{C}}"'] \arrow[dr, "\iota^{\mathcal{D}}"] & & \\
& f\Phi^{\mathcal{C}}(x) \arrow[ul, "f\iota^{\mathcal{C}}"] & & f\Phi^{\mathcal{C}}\mu^{\mathcal{C}}(1^{\mathcal{C}},x) \arrow[ll, "f\Phi^{\mathcal{C}}\lambda^{\mathcal{C}}"] & f\Phi^{\mathcal{D}}\mu^{\mathcal{C}}(\Phi^{\mathcal{C}}(1^{\mathcal{C}}),x) \arrow[l, "f\mathcal{L}^{\mathcal{C}}"]
\end{tikzcd}$$

wherein each interior sub-diagram commutes:

- the top left square commutes by naturality of $\mathcal{L}^{\mathcal{D}}$;
- the square below it is the image of left unitality axiom of Definition 4.1 for $f_\bullet$ under $\Phi^{\mathcal{D}}$;
- the trapezoid beneath that is the image of the naturality square of $\iota^{\mathcal{C}}$ under $\Phi^{\mathcal{D}}f$;
- the parallelogram at the bottom left commutes by the naturality of $\iota^{\mathcal{C}}$;
- the trapezoid to its left commutes by the naturality of $\iota^{\mathcal{D}}$;

– the triangle to the right of this is the coherence axiom (4.13) of ξ with ι;
– and the heptagon on the right is the coherence axiom (4.11) of ξ with the left absorbers.

Commutativity of the remaining diagram, witnessing the compatibility of $(\zeta^{\mathrm{Int}}, \nu^{\mathrm{Int}})$ with the right unitors, follows from a symmetric diagram chase. Hence we conclude that $\mathrm{Int}(f_\bullet) : \mathrm{Int}(\mathcal{C}_\bullet) \to \mathrm{Int}(\mathcal{D}_\bullet)$ is monoidal. □

Proposition 4.14 suggests a functorial interpretation of Int, which we make formal shortly. First, we must detour briefly to describe the remaining 2-categorical structure on internalized categories.

Each internalized category $(\mathcal{C}_\bullet, \Phi^{\mathcal{C}})$ is endowed with an internalized functor whose underlying graded monoidal functor is the identity, which we also denote simply by $\mathrm{id}_{\mathcal{C}_\bullet}$. The structure maps ξ for $\mathrm{id}_{\mathcal{C}_\bullet}$ are simply the identity componentwise, and the axioms are trivial. Furthermore, given a pair of internalized functors

$$(\mathcal{C}_\bullet, \Phi^{\mathcal{C}}) \xrightarrow{f_\bullet} (\mathcal{D}_\bullet, \Phi^{\mathcal{D}}) \xrightarrow{g_\bullet} (\mathcal{E}_\bullet, \Phi^{\mathcal{E}})$$

there is an internalized functor $(gf)_\bullet : (\mathcal{C}_\bullet, \Phi^{\mathcal{C}}) \to (\mathcal{E}_\bullet, \Phi^{\mathcal{E}})$ whose underlying graded monoidal functor is the graded monoidal composite of $f_\bullet$ with $g_\bullet$. As one might guess, the structure maps are given by the composites

$$\Phi^{\mathcal{E}}_m g_m f_m \Rightarrow g_1 \Phi^{\mathcal{D}}_m f_m \Rightarrow g_1 f_1 \Phi^{\mathcal{C}}_m,$$

which again can be shown to satisfy the coherence conditions after some amount of tedious diagram chasing.

Definition 4.15. With the notions of identity and composition sketched above, we obtain a category **IntMonCat** of internalized graded monoidal categories and internalized functors between them.

Of course, we are now compelled to define a suitable notion of natural transformation between internalized functors, to fulfill the prophecy that **IntMonCat** forms a 2-category.

Definition 4.16. Let $f_\bullet, g_\bullet : (\mathcal{C}_\bullet, \Phi^{\mathcal{C}}) \to (\mathcal{D}_\bullet, \Phi^{\mathcal{D}})$ be internalized functors. An *internalized (graded monoidal natural) transformation* $\beta_\bullet : f_\bullet \Rightarrow g_\bullet$ is a graded monoidal transformation $f_\bullet \Rightarrow g_\bullet$ such that, for each $m \in \mathbb{N}$, the following square commutes:

$$\begin{tikzcd}
\Phi^{\mathcal{D}}_m f_m \arrow[r, Rightarrow, "\Phi^{\mathcal{D}}_m \beta_m"] \arrow[d, Rightarrow, "\xi^m"'] & \Phi^{\mathcal{D}}_m g_m \arrow[d, Rightarrow, "\xi^m"] \\
f_1 \Phi^{\mathcal{C}}_1 \arrow[r, Rightarrow, "\beta_1 \Phi^{\mathcal{C}}_1"'] & g_1 \Phi^{\mathcal{C}}_1
\end{tikzcd}$$

Just as internalized functors internalized to monoidal functors, internalized transformations likewise internalize to monoidal transformations:

Proposition 4.17. Let $\beta_\bullet : f_\bullet \Rightarrow g_\bullet$ be an internalized transformation between internalized functors $f_\bullet, g_\bullet : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$. Then $\mathrm{Int}(\beta) := \beta_1 : \mathrm{Int}(f) \Rightarrow \mathrm{Int}(g)$ defines a monoidal transformation between the monoidal functors.

Proof. To show that β_1 is monoidal, we must verify that the diagrams

$$\begin{tikzcd}
& 1^{\mathcal{D}} \arrow[dl, "\nu^{\mathrm{Int}}"'] \arrow[dr, "\nu^{\mathrm{Int}}"] & \\
f_1 \Phi^{\mathcal{C}}_0(1^{\mathcal{C}}) \arrow[rr, "\beta_1"'] & & g_1 \Phi^{\mathcal{C}}_0(1^{\mathcal{C}})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Phi_2^{\mathcal{D}}\mu^{\mathcal{D}}(f_1(x), f_1(y)) \arrow[r, "\Phi_2^{\mathcal{D}}\mu^{\mathcal{D}}(\beta_1,\beta_1)"] \arrow[d, "\zeta^{\mathrm{Int}}"'] & \Phi_2^{\mathcal{D}}\mu^{\mathcal{D}}(g_1(x), g_1(y)) \arrow[d, "\zeta^{\mathrm{Int}}"'] \\
f_1\Phi_2^{\mathcal{C}}\mu^{\mathcal{C}}(x,y) \arrow[r, "\beta_1"'] & g_1\Phi_2^{\mathcal{C}}\mu^{\mathcal{C}}(x,y)
\end{tikzcd}$$

commute (for all $x, y \in \mathcal{C}_1$). Unfolding the definitions from Proposition 4.14, we can rewrite the first diagram as the perimeter of

$$\begin{tikzcd}
& 1^{\mathcal{D}} \arrow[dl, "\Phi_0^{\mathcal{D}}\nu"'] \arrow[dr, "\Phi_0^{\mathcal{D}}\nu"] & \\
\Phi_0^{\mathcal{D}}f_0(1^{\mathcal{C}}) \arrow[rr, "\Phi_0^{\mathcal{D}}\beta_0"'] \arrow[d, "\xi^0"'] & & \Phi_0^{\mathcal{D}}g_0(1^{\mathcal{C}}) \arrow[d, "\xi^0"] \\
f_1\Phi_0^{\mathcal{C}}(1^{\mathcal{C}}) \arrow[rr, "\beta_1"'] & & g_1\Phi_0^{\mathcal{C}}(1^{\mathcal{C}})
\end{tikzcd} \tag{4.18}$$

and the second as the perimeter of

$$\begin{tikzcd}
\Phi_2^{\mathcal{D}}\mu^{\mathcal{D}}(f_1(x), f_1(y)) \arrow[r, "\Phi_2^{\mathcal{D}}\mu^{\mathcal{D}}(\beta_1,\beta_1)"] \arrow[d, "\Phi_2^{\mathcal{D}}\zeta^{1,1}"'] & \Phi_2^{\mathcal{D}}\mu^{\mathcal{D}}(g_1(x), g_1(y)) \arrow[d, "\Phi_2^{\mathcal{D}}\zeta^{1,1}"] \\
\Phi_2^{\mathcal{D}}f_2\mu^{\mathcal{C}}(x,y) \arrow[r, "\Phi_2^{\mathcal{D}}\beta_2"'] \arrow[d, "\zeta^2"'] & \Phi_2^{\mathcal{D}}g_2\mu^{\mathcal{C}}(x,y) \arrow[d, "\zeta^2"] \\
f_1\Phi_2^{\mathcal{C}}\mu^{\mathcal{C}}(x,y) \arrow[r, "\beta_1"'] & g_1\Phi_2^{\mathcal{C}}\mu^{\mathcal{C}}(x,y)
\end{tikzcd} \tag{4.19}$$

Both diagrams commute because each of their interior subdiagrams commute:

(1) In diagram (4.18) the top triangle is the image under $\Phi_0^{\mathcal{D}}$ of the coherence of $\beta_\bullet$ with the structure maps ν, and the bottom square is the coherence of $\beta_\bullet$ with ξ^0.

(2) In diagram (4.19) the top square is the image under $\Phi_2^{\mathcal{D}}$ of the coherence of $\beta_\bullet$ with $\zeta^{1,1}$, and the bottom square is the coherence of $\beta_\bullet$ with ξ^2.

It follows that $\mathrm{Int}(\beta)$ is a monoidal transformation $\mathrm{Int}(f) \Rightarrow \mathrm{Int}(g)$. □

Having defined $\mathrm{Int}(-)$ on internalized categories, functors, and natural transformations, we are finally ready to state the following:

Proposition 4.20. Internalization defines a 2-functor

$$\mathrm{Int} : \mathbf{IntMonCat} \to \mathbf{MonCat},$$

which is the restriction of $\mathrm{Forget}_{\mathrm{Gr}}$ along the forgetful functors for monoidal categories and internalized categories. That is, the square of 2-categories

$$\begin{tikzcd}
\mathbf{IntMonCat} \arrow[r, "\mathrm{Int}"] \arrow[d, "\mathrm{Forget}_{\mathrm{Int}}"'] & \mathbf{MonCat} \arrow[d, "\mathrm{Forget}_{\mathrm{Mon}}"] \\
\mathbf{GrMonCat} \arrow[r, "\mathrm{Forget}_{\mathrm{Gr}}"'] & \mathbf{Cat}
\end{tikzcd}$$

commutes.

Proof. The action of Int on objects is clear, while the actions on 1-morphisms and 2-morphisms is defined in Propositions 4.14 and 4.17 (where we already verified that these constructions satisfy the required conditions). Functoriality is automatic, since the composition and identities in **IntMonCat** restrict to classical functor composition and identities in degree 1.

Commutativity of the diagram is likewise straightforward: the underlying category (resp. functor, natural transformation) of an internalized category $\mathcal{C}_\bullet$ (resp. internalized functor $f_\bullet$, internalized transformation $\beta_\bullet$) is simply the degree 1 component $\mathcal{C}$ (resp. f_1, β_1). But the composite $\mathrm{Forget}_{\mathrm{Gr}} \circ \mathrm{Forget}_{\mathrm{Int}}$ likewise isolates the degree 1 data of an internalized category, by first forgetting the internalization and then concentrating in degree 1, so the desired square commutes. □

Next, we make formal the property that internalizing the free internalized category of a monoidal category recovers the original monoidal structure:

Theorem 4.21. Int has a section Free : **MonCat** → **IntMonCat**, which is the restriction of Free : **Cat** → **GrMonCat** along the forgetful functors for monoidal categories and internalized categories. That is, the pair of squares of 2-categories

$$
\begin{tikzcd}
\mathbf{MonCat} \arrow[r, "\mathrm{Free}"] \arrow[d, "\mathrm{Forget}_{\mathrm{Mon}}"'] & \mathbf{IntMonCat} \arrow[d, "\mathrm{Forget}_{\mathrm{Int}}"] \\
\mathbf{Cat} \arrow[r, "\mathrm{Free}_{\mathrm{Gr}}"'] & \mathbf{GrMonCat}
\end{tikzcd}
\tag{4.22}
$$

commute.

Proof. For the diagram (4.22) to commute, for a monoidal category $(\mathcal{C}, \otimes, I)$ we are forced to take $\mathrm{Free}_{\mathrm{Gr}}(\mathcal{C})$ the underlying graded monoidal category of $\mathrm{Free}(\mathcal{C})$. We equip this graded monoidal category with the internalization $\Phi_\bullet^{\mathrm{Free}}$, where $\Phi_0^{\mathrm{Free}} : 1 \to \mathcal{C}$ identifies the unit $I \in \mathcal{C}$, Φ_1^{Free} is the identity functor on $\mathcal{C}$, and

$$\Phi_m^{\mathrm{Free}} : \mathcal{C}^m \to \mathcal{C}$$
$$\Phi_m^{\mathrm{Free}}(x_1, \dots, x_m) := \bigotimes_{i=1}^{m} x_i$$

for all $m > 1$ (fixing a bias for $\otimes$, as usual). The absorbers of $\Phi_\bullet^{\mathrm{Free}}$ are induced by the monoidal structure of $\otimes$: if $m, n > 0$ then $\mathcal{L}_{m,n}$ is the (formal) composite [6] of the associators of $\otimes$ witnessing the isomorphism

$$x_1 \otimes \cdots \otimes x_m \otimes \left(\bigotimes_{j=m+1}^{m+n} x_j \right) \cong \bigotimes_{i=1}^{m+n} x_i$$

and $\mathcal{L}_{0,n}$ is supplied via the left unitor

$$I \otimes \left(\bigotimes_{i=n}^{n} x_i \right) \cong \bigotimes_{i=1}^{n} x_i.$$

$\mathcal{R}$ is defined similarly, and by Mac Lane's Coherence Theorem these absorbers satisfy the axioms of Definition 3.18, making $(\mathrm{Free}(\mathcal{C})_\bullet, \Phi_\bullet^{\mathrm{Free}})$ an internalized category.

Given a monoidal functor $f : (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}})$, our definition of the underlying graded monoidal functor $\mathrm{Free}(f)_\bullet : \mathrm{Free}(\mathcal{C})_\bullet \to \mathrm{Free}(\mathcal{D})_\bullet$ is once again forced by (4.22),

$$\mathrm{Free}(f)_\bullet := \mathrm{Free}_{\mathrm{Gr}}(f)_\bullet.$$

To extend this construction to a functor on (free) internalized categories, we must specify natural structure maps

$$\xi^m : \Phi_m^{\mathrm{Free}(\mathcal{D})_\bullet} \mathrm{Free}(f)_m \Rightarrow f\Phi_m^{\mathrm{Free}(\mathcal{C})_\bullet},$$

which, by unwinding the definitions, consist merely of maps

$$\bigotimes_{i=1}^{m} f(x_i) \to f\left(\bigotimes_{i=1}^{m} x_i\right)$$

for each $x_1, \ldots, x_m \in \mathcal{C}$ (where the empty tensor product is the unit). These are provided by the structure maps

$$\begin{aligned} \zeta_{x,y} &: f(x) \otimes_{\mathcal{D}} f(y) \to f(x \otimes_{\mathcal{C}} y) \\ \nu &: I_{\mathcal{D}} \to f(I_{\mathcal{C}}) \end{aligned}$$

(where the maps ζ must be iterated consistently with the fixed choice of parenthesization for $\otimes$). The necessary diagrams commute by appealing to the Coherence Theorem (for lax monoidal functors – see [6, Section XI.2]), using the axioms of ζ and ν as the structure maps of a monoidal functor to push the associators and unitors through them.

Finally, for a monoidal transformation $\beta : f \Rightarrow g$ between parallel monoidal functors, we again must take the graded monoidal transformation underlying $\mathrm{Free}(\beta)$ to be $\mathrm{Free}_{\mathrm{Gr}}(\beta)$. The condition of Definition 4.16 can be rewritten as

$$\begin{tikzcd} \bigotimes_{i=1}^m f(x_i) \arrow[r, "\otimes_i \beta"] \arrow[d, "\xi^m"'] & \bigotimes_{i=1}^m g(x_i) \arrow[d, "\xi^m"] \\ f\left(\bigotimes_{i=1}^m x_i\right) \arrow[r, "\beta"'] & g\left(\bigotimes_{i=1}^m x_i\right) \end{tikzcd}$$

commuting for each $x_1, \ldots, x_m \in \mathcal{C}$. Since β is a monoidal transformation and ξ^m is defined using the structure maps ζ and ν, each such square indeed commutes, completing the construction of $\mathrm{Free} : \mathbf{MonCat} \to \mathbf{IntMonCat}$. 2-functoriality is immediate, since the categorical structures on **MonCat** and **IntMonCat** are compatible and $\mathrm{Free}(-)$ is defined componentwise in every dimension.

Throughout our construction of $\mathrm{Free}(-)$ above, we took care to guarantee that the square (4.22) commutes, so it remains to be shown that $\mathrm{Int} \circ \mathrm{Free} = \mathrm{id}_{\mathbf{MonCat}}$. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. On underlying categories we have

$$\mathrm{Int}(\mathrm{Free}(\mathcal{C})) = \mathrm{Free}(\mathcal{C})_1 = \mathcal{C}^1 = \mathcal{C},$$

and given objects $x, y \in \mathcal{C}$ the internalized monoidal product of x and y is

$$\Phi_2 \mu_{1,1}(x, y) = \Phi_2(x, y) = x \otimes y.$$

The same argument holds for morphisms in $\mathcal{C}$, and the associator and unitors for the internalized product agree with those of $\otimes$ by the Coherence Theorem for $(\mathcal{C}, \otimes, I)$, since the the free monoidal product is strict, and the structure maps for the internalized product are formal composites of the structure maps of $\otimes$. Thus, Free is indeed a right inverse for Int, completing the proof. $\square$

By virtually the same proof as for graded monoidal categories, we obtain an adjunction Free $\dashv$ Int embedding monoidal categories in internalized categories.

Theorem 4.23. Free is left adjoint to Int, and embeds **MonCat** as a coreflective subcategory of **IntMonCat**.

Proof. The definitions of the unit $\eta : \mathrm{id}_{\mathbf{MonCat}} \Rightarrow \mathrm{Int} \circ \mathrm{Free}$ and $\epsilon : \mathrm{Free} \circ \mathrm{Int} \Rightarrow \mathrm{id}_{\mathbf{IntMonCat}}$ are identical to the constructions of Proposition 4.8, and since $\mathrm{Int} \circ \mathrm{Free} = \mathrm{id}_{\mathbf{MonCat}}$ by Theorem 4.21 the same argument shows that this lifts to an adjunction Free $\dashv$ Int. As before, since Int is a retraction of Free the latter is an embedding, and since η is the identity componentwise it follows that Free embeds **MonCat** as a coreflective subcategory. $\square$

Corollary 4.24. Internalization preserves all limits in **IntMonCat** (as a 1-category).

Proof. Immediate from Theorem 4.23. $\square$

5. Internal Smash Products of Spectra

We conclude this paper by outlining two applications of internalized categories to reconstruct a pair of monoidal products which occur naturally in stable homotopy theory.

5.1. Day Convolution.

The smash product of symmetric or orthogonal spectra (or, more generally, of diagram spectra over a given monoid object in the category of $\mathcal{D}$-spaces, where $\mathcal{D}$ is some topological category – see [8]) can be formulated as a special case of Day convolution. In this section we describe how Day convolution fits into the framework of an internalized graded monoidal category, and use this to prove that the Day convolution monoidal product is closed and symmetric.

We first recall the general procedure of Day convolution. We fix a bicomplete, closed, symmetric monoidal category $(\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}})$, and a (small) $\mathcal{V}$-enriched monoidal category $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$. We additionally denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{V})$ the $\mathcal{V}$-enriched functor category, with hom-object $\mathrm{Fun}(\mathcal{C}, \mathcal{V})(F, G)$ defined by the $\mathcal{V}$-enriched end

$$\mathrm{Fun}(\mathcal{C}, \mathcal{V})(F, G) := \int_{c \in \mathcal{C}} \underline{V}(Fc, Gc)$$

for given $\mathcal{V}$-functors $F, G : \mathcal{C} \to \mathcal{V}$. (Readers unfamiliar with this notation or who wish to review (co)ends should refer to Section 1.2 of [9] or Section 1.1.3 of [4].) Lastly, given $\mathcal{V}$-functors $F : \mathcal{C} \to \mathcal{A}$ and $K : \mathcal{C} \to \mathcal{B}$ (where $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories), we write $\mathrm{Lan}_K(F)$ for the left Kan extension $\mathcal{B} \to \mathcal{A}$ of F along K, as defined in Section 4.1 of [5].

Definition 5.1. The *Day convolution product* on $\mathrm{Fun}(\mathcal{C}, \mathcal{V})$ is the bifunctor

$$\otimes_{\mathrm{Day}} : \mathrm{Fun}(\mathcal{C}, \mathcal{V}) \times \mathrm{Fun}(\mathcal{C}, \mathcal{V}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{V})$$

defined by the left Kan extension

$$F \otimes_{\mathrm{Day}} G := \mathrm{Lan}_{\otimes_{\mathcal{C}}}(F \otimes_{\mathcal{V}} G)$$

for $\mathcal{V}$-functors $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{V})$, where

$$(F \otimes_{\mathcal{V}} G)(c) := Fc \otimes_{\mathcal{V}} Gc.$$

By way of the coend formula for left Kan extensions (see (4.25) in [5]), we compute the image of $c \in \mathcal{C}$ under $F \otimes_{\mathrm{Day}} G$ as

$$(F \otimes_{\mathrm{Day}} G)(c) = \int^{c_1, c_2 \in \mathcal{C}} \mathcal{C}(c_1 \otimes_{\mathcal{C}} c_2, c) \otimes_{\mathcal{V}} Fc_1 \otimes_{\mathcal{V}} Gc_2.$$

Remark 5.2. The product $\otimes_{\mathrm{Day}}$ is indeed monoidal, with unit object given by $\mathcal{Y}(I_{\mathcal{C}}) : \mathcal{C} \to \mathcal{V}$, where $\mathcal{Y}$ is the $\mathcal{V}$-enriched (contravariant) Yoneda embedding $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{V})$. One can prove this classically, but this will also follow immediately from Theorem 3.25 after we've shown that $\otimes_{\mathrm{Day}}$ arises as the internalization of a graded monoidal product.

Definition 5.3. We define a $\mathcal{V}$-enriched graded monoidal category $(\mathrm{Fun}_\bullet(\mathcal{C}, \mathcal{V}), \mu, 1)$ by

1. $\mathrm{Fun}_m(\mathcal{C}, \mathcal{V}) := \mathrm{Fun}(\mathcal{C}^m, \mathcal{V})$ for $m \in \mathbb{N}$.
2. $\mu_{m,n}(F, G) : \mathcal{C}^{m+n} \to \mathcal{V}$ is the $\mathcal{V}$-functor
$$\mu_{m,n}(F, G)(c_1, \dots, c_{m+n}) := F(c_1, \dots, c_m) \otimes_V G(c_{m+1}, \dots, c_{m+n})$$
for $m, n \in \mathbb{N}$, where $F : \mathcal{C}^m \to \mathcal{V}$ and $G : \mathcal{C}^n \to \mathcal{V}$ are $\mathcal{V}$-functors.
3. $1 : \mathcal{C}^0 \to \mathcal{V}$ is the functor which carries the terminal $\mathcal{V}$-category $\mathcal{C}^0$ to the object $I_{\mathcal{V}} \in \mathcal{V}$.

The natural isomorphisms for μ are defined termwise by the corresponding isomorphisms for $\otimes_{\mathcal{V}}$.

We aim to show that $\otimes_{\mathrm{Day}}$ is an internalization of μ:

Definition 5.4. Given $m \in \mathbb{N}$, let $\Phi_m : \mathrm{Fun}(\mathcal{C}^m, \mathcal{V}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{V})$ be the $\mathcal{V}$-functor
$$\Phi_m(F) := \mathrm{Lan}_{\otimes_{\mathcal{C}}^m}(F),$$
where $\otimes_{\mathcal{C}}^m : \mathcal{C}^m \to \mathcal{C}$ is the[7] $\mathcal{V}$-functor
$$(c_1, \dots, c_m) \mapsto \bigotimes_{i=1}^{m} c_i.$$
In the special case that $m = 0$, we adopt the convention that $\otimes_{\mathcal{C}}^0$ is the $\mathcal{V}$-functor $\mathcal{C}^0 \to \mathcal{C}$ which identifies the unit $I_{\mathcal{C}}$.

Remark 5.5. We note that since $\mathrm{Fun}_0(\mathcal{C}, \mathcal{V}) = \mathrm{Fun}(\mathcal{C}^0, \mathcal{V})$ is naturally isomorphic to $\mathcal{V}$, Φ_0 can be viewed as a $\mathcal{V}$-functor $\mathcal{V} \to \mathrm{Fun}(\mathcal{C}, \mathcal{V})$. Given $v \in \mathcal{V}$ and $c \in \mathcal{C}$, we have
$$\Phi_0(v)(c) = (\mathrm{Lan}_{\otimes_{\mathcal{C}}^0}(v))(c) = \int^{\mathcal{C}^0} \mathcal{C}(I_{\mathcal{C}}, c) \otimes_{\mathcal{V}} v = \mathcal{Y}(I_{\mathcal{C}})(c) \otimes_{\mathcal{V}} v$$
by the coend formula for left Kan extensions.

Theorem 5.6. Φ defines an internalization on $(\mathrm{Fun}_\bullet(\mathcal{C}, \mathcal{V}), \mu, 1)$.

Proof. First, we observe that $\Phi_1(F)$ is the left Kan extension of F along $\mathrm{id}_{\mathcal{C}}$, so there is a natural isomorphism $u : \Phi_1 \cong \mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{V})}$.

Next, we will construct the left absorber for Φ as the composite of a chain of natural isomorphisms; the construction of the right absorber is completely dual.

Fix $m, n \in \mathbb{N}$, and let $F : \mathcal{C}^m \to \mathcal{V}$ and $G : \mathcal{C}^n \to \mathcal{V}$ be $\mathcal{V}$-functors. Additionally, let $c \in \mathcal{C}$. We aim to construct a (natural) isomorphism
$$\mathrm{Lan}_{\otimes_{\mathcal{C}}^{n+1}}(\mathrm{Lan}_{\otimes_{\mathcal{C}}^m}(F), G)(c) \cong \mathrm{Lan}_{\otimes_{\mathcal{C}}^{m+n}}(F, G)(c).$$

[7]Here one should be slightly careful in the case that $\otimes_{\mathcal{C}}$ is not strict. One is then forced to choose a particular arrangement of parentheses, that is, a bias for the m-fold monoidal product on $\mathcal{C}$. Each natural isomorphism in the internalization is then composed with some sequence of associators to ensure compatibility, but Mac Lane's Coherence Theorem guarantees that this will not cause problems.

Unwinding the left-hand side using the coend formula, we obtain

$$\int^{d,c_1,\ldots,c_n} \mathcal{C}\Big(d\otimes_{\mathcal{C}}\bigotimes_{i=1}^{n} c_i, c\Big)\otimes_{\mathcal{V}}\left(\int^{d_1,\ldots,d_m} \mathcal{C}\Big(\bigotimes_{j=1}^{m} d_j, d\Big)\otimes_{\mathcal{V}} F(d_1,\ldots,d_m)\right)\otimes_{\mathcal{V}} G(c_1,\ldots,c_n),$$

where both coends are indexed by $\mathcal{C}$. By the Fubini Theorem, this is naturally isomorphic to

$$\int^{d,d_1,\ldots,d_m,c_1,\ldots,c_n} \mathcal{C}\Big(d\otimes_{\mathcal{C}}\bigotimes_{i=1}^{n} c_i, c\Big)\otimes_{\mathcal{V}} \mathcal{C}\Big(\bigotimes_{j=1}^{m} d_j, d\Big)\otimes_{\mathcal{V}} F(d_1,\ldots,d_m)\otimes_{\mathcal{V}} G(c_1,\ldots,c_n),$$

which is naturally isomorphic to

$$\int^{d_1,\ldots,d_m,c_1,\ldots,c_n} \mathcal{C}\Big(\bigotimes_{j=1}^{m} d_j\otimes_{\mathcal{C}}\bigotimes_{i=1}^{n} c_i, c\Big)\otimes_{\mathcal{V}} F(d_1,\ldots,d_m)\otimes_{\mathcal{V}} G(c_1,\ldots,c_n)$$

via the co-Yoneda Lemma. But using the definition of Φ and the coend formula for left Kan extensions, this is precisely

$$\mathrm{Lan}_{\otimes_{\mathcal{C}}^{m+n}}(F,G)(c),$$

so the desired natural isomorphism is the composite of the isomorphisms above. Note that in the case that one of m or n is 0 the same argument holds, with the corresponding m-fold or n-fold tensor products being replaced by the unit object $I_{\mathcal{C}}$.

In the interest of preserving the reader's (and author's) sanity, we omit the verification of the coherence axioms for the absorbers. □

The following is then immediate from Theorem 3.25 and Definition 5.1:

Proposition 5.7. The monoidal product on $\mathrm{Fun}(\mathcal{C},\mathcal{V})$ induced by Φ is the Day convolution product $\otimes_{\mathrm{Day}}$.

Corollary 5.8. The Day convolution product is a monoidal product, with unit $\mathcal{Y}(I_{\mathcal{C}})$.

Proof. It follows from Proposition 5.7 and Theorem 3.25 that $\otimes_{\mathrm{Day}}$ is a monoidal product, while the second claim is a consequence of Remark 5.5, since

$$\Phi_0(1)(c) = \mathcal{Y}(I_{\mathcal{C}})(c)\otimes_{\mathcal{V}} I_{\mathcal{V}} \cong \mathcal{Y}(I_{\mathcal{C}})(c)$$

for each $c\in\mathcal{C}$. □

We have shown that Day convolution is a special case of internalization. In fact, we can further recover the properties of symmetry and closure that $\otimes_{\mathrm{Day}}$ enjoys by using the formal behavior of internalized categories.

Lemma 5.9. The graded monoidal category $\mathrm{Fun}_\bullet(\mathcal{C},\mathcal{V})$ is symmetric, as is its internalization (see Definition 3.10 and Definition 3.28).

Proof. This follows from the symmetry of $\otimes_{\mathcal{V}}$, and a diagram chase on its braiding and the absorbers of Φ. □

Lemma 5.10. The graded monoidal category $\mathrm{Fun}_\bullet(\mathcal{C},\mathcal{V})$ is closed (see Definition 3.14).

Proof. We define the internal hom of μ by the $\mathcal{V}$-functors

$$\mathscr{F}_{m,n} : \mathrm{Fun}(\mathcal{C}^n,\mathcal{V})^{\mathrm{op}}\times\mathrm{Fun}(\mathcal{C}^{m+n},\mathcal{V})\to\mathrm{Fun}(\mathcal{C}^m,\mathcal{V})$$

by

$$\mathscr{F}_{m,n}(F,G)(c_1,\ldots,c_m) := \int_{d_1,\ldots,d_n\in\mathcal{C}} \underline{V}(F(d_1,\ldots,d_n), G(c_1,\ldots,c_m,d_1,\ldots,d_n)),$$

where $c_1,\ldots,c_m \in \mathcal{C}$, $F : \mathcal{C}^n \to \mathcal{V}$, and $G : \mathcal{C}^{m+n} \to \mathcal{V}$. To verify that this does indeed define an internal hom of μ, fix $m,n \in \mathbb{N}$, and suppose given $\mathcal{V}$-functors $F : \mathcal{C}^m \to \mathcal{V}$, $G : \mathcal{C}^n \to \mathcal{V}$, and $H : \mathcal{C}^{m+n} \to \mathcal{V}$. Then, unwinding the definitions, we have

$$\mathrm{Fun}_{m+n}(\mathcal{C},\mathcal{V})(\mu(F,G),H) = \int_{c_1,\ldots,c_{m+n}\in\mathcal{C}} \underline{V}(F(c_1,\ldots,c_m)\otimes_{\mathcal{V}} G(c_{m+1},\ldots,c_{m+n}), H(c_1,\ldots,c_{m+n})),$$

and the right-hand side is isomorphic to

$$\int_{c_1,\ldots,c_{m+n}\in\mathcal{C}} \underline{V}(F(c_1,\ldots,c_m), \underline{V}(G(c_{m+1},\ldots,c_{m+n}), H(c_1,\ldots,c_{m+n})))$$

using the closed structure of $\mathcal{V}$. Now, using the universal property of ends, this is isomorphic to

$$\int_{c_1,\ldots,c_m\in\mathcal{C}} \underline{V}(F(c_1,\ldots,c_m), \int_{c_{m+1},\ldots,c_{m+n}} \underline{V}(G(c_{m+1},\ldots,c_{m+n}), H(c_1,\ldots,c_{m+n}))),$$

which is precisely

$$\mathrm{Fun}_m(\mathcal{C},\mathcal{V})(F,\mathscr{F}(G,H)),$$

and hence μ is closed with internal hom $\mathscr{F}$. □

Lemma 5.11. The internalization Φ of μ has a right adjoint, and is closed (see Definition 3.33).

Proof. Since μ is closed by Lemma 5.10, the second claim follows immediately from the first. Given $m \in \mathbb{N}$ and a $\mathcal{V}$-functor $G : \mathcal{C}^m \to \mathcal{V}$, we define a $\mathcal{V}$-functor $\Psi_m : \mathrm{Fun}(\mathcal{C},\mathcal{V}) \to \mathrm{Fun}(\mathcal{C}^m,\mathcal{V})$ by

$$\Psi_m(G)(c_1,\ldots,c_m) := G\Big(\bigotimes_{i=1}^{m} c_i\Big)$$

for each $c_1,\ldots,c_m \in \mathcal{C}$. Then since Φ_m is given by left Kan extension along $\otimes_{\mathcal{C}}^m$, and Ψ_m precomposes with $\otimes_{\mathcal{C}}^m$, it is automatic that Φ_m is left adjoint to Ψ_m. □

Proposition 5.12. The Day convolution product is symmetric and closed.

Proof. Symmetry follows from Corollary 3.29 and Lemma 5.10, and closure follows from Corollary 3.34, Lemma 5.10, and Lemma 5.11. □

To specialize to the case of spectra (symmetric, orthogonal, etc.) we refer to [8]. Fix a symmetric monoidal category $\mathcal{D}$ enriched over based topological spaces, and recall that a *$\mathcal{D}$-space* is a continuous functor $\mathcal{D} \to \mathbf{Top}_*$. Taking $\mathcal{V} := \mathbf{Top}_*$, with symmetric monoidal product given by the usual smash product $\wedge$, and $\mathcal{C} := \mathcal{D}$, we see that $\mathrm{Fun}(\mathcal{C},\mathcal{V})$ is precisely the category $\mathcal{D}(\mathbf{Top}_*)$ of $\mathcal{D}$-spaces. Day convolution then provides a means of turning the external product induced by $\wedge$ into the *internal smash product of $\mathcal{D}$-spaces* – see Definition 21.4 of [8].

A *$\mathcal{D}$-spectrum* (Definition 1.9 in [8]) over a monoid object R in $\mathcal{D}(\mathbf{Top}_*)$ is a $\mathcal{D}$-space equipped with a right action by R. These form a category $\mathbf{Spec}(\mathcal{D})_R$ which, by Theorem 2.2 of [8], is equivalent to the category of $\mathcal{D}_R$-spaces, where $\mathcal{D}_R$ is a $\mathbf{Top}_*$-enriched category defined from $\mathcal{D}$ and R (see Construction 2.1 of [8] for a complete definition). Moreover, when R is a commutative monoid $\mathcal{D}_R$ is naturally equipped with a symmetric monoidal product, in which case Day convolution induces a symmetric monoidal product on the category of $\mathcal{D}$-spectra.

Finally, we note that both symmetric spectra and orthogonal spectra can be realized as $\mathcal{D}$-spectra over commutative monoids. In the case of the former (see Example 4.2 of [8]), we have $\mathcal{D} := \Sigma$, the category of (unbased) finite sets $[n] := \{0, \dots, n\}$ for each $n \in \mathbb{N}$ and their permutations, and $R := S_\Sigma$, the continuous functor $\Sigma \to \mathbf{Top}_*$ which sends $[n]$ to the n-sphere S^n. For the latter (Example 4.4 of [8]) we instead take $\mathcal{D}$ to be the category of (unbased) finite-dimensional real inner product spaces and isometric linear isomorphisms between them, and let R be the functor $\mathcal{D} \to \mathbf{Top}_*$ sending V to the one-point compactification S^V. In both cases, the resulting internal smash product induced by Day convolution is precisely the usual smash product of the corresponding spectra.

5.2. **The Smash Product of S-Modules.** Finally, we turn our attention to S-modules, defined in [1]. After reviewing the relevant definitions, we will show how the smash product of S-modules can be realized as an internalization of the external smash product of coordinate-free spectra.

Recall that, given a real inner product U isomorphic to $\mathbb{R}^\infty$, the direct sum of countably-many copies of $\mathbb{R}$, there exists a category $\mathscr{S}U$ of *coordinate-free spectra over the universe* U. Throughout the remainder of this section, we will refer to these as "spectra (over U)."

The category $\mathscr{S}U$ enjoys a number of convenient properties. By Proposition I.1.1 in [1] it is both complete and cocomplete. Moreover, given a pair of universes U and U', there is an *external smash product*

$$\wedge : \mathscr{S}U \times \mathscr{S}U' \to \mathscr{S}(U \oplus U')$$

of spectra over U and U', which is associative and symmetric (up to natural isomorphism). Lastly, there is a functor $\Sigma^\infty : \mathbf{Top}_* \to \mathscr{S}U$, called the *suspension spectrum functor*, which identifies a based space with a spectrum over U, and we write S to denote the suspension spectrum $\Sigma^\infty S^0$.

For universes U and U', let $\mathscr{I}(U, U')$ be the set of linear isometries $U \to U'$. Viewing U and U' as topological spaces, whose topologies are induced by their finite dimensional subspaces, we can equip $\mathscr{I}(U, U')$ with the function space topology. Now, given any space A and a continuous map $\alpha : A \to \mathscr{I}(U, U')$, together with a spectrum E over U, there exists a spectrum $A \ltimes E$ over U', called the *twisted half-smash product* of A and E. Moreover, this construction is functorial in E, and satisfies the following properties (see Proposition I.2.2 of [1]):

(1) For each $E \in \mathscr{S}U$ there is a canonical isomorphism
$$\{\mathrm{id}_U\} \ltimes E \cong E.$$
(2) For spaces A, B with maps $\alpha : A \to \mathscr{I}(U, U')$ and $\beta : B \to \mathscr{I}(U', U'')$ and a spectrum $E \in \mathscr{S}U$ there is a canonical isomorphism
$$(B \times A) \ltimes E \cong B \ltimes (A \ltimes E).$$
(3) Let A_1 and A_2 be spaces with maps $\alpha_i : A_i \to \mathscr{I}(U_i, U_i')$. Then given spectra $E_1 \in \mathscr{S}U_1$ and $E_2 \in \mathscr{S}U_2$ there is a canonical isomorphism
$$(A_1 \times A_2) \ltimes (E_1 \wedge E_2) \cong (A_1 \ltimes E_1) \wedge (A_2 \ltimes E_2).$$
(4) For each $E \in \mathscr{S}U$ and each based space X there is a canonical isomorphism
$$A \ltimes (E \wedge X) \cong (A \ltimes E) \wedge X.$$

Now, writing U^j for the direct sum of j copies of a universe U, and taking $\mathscr{L}(j) := \mathscr{I}(U^j, U)$, there is a (symmetric) operad, called the *linear isometries operad*, equipped with structure maps

$$\gamma : \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \to \mathscr{L}(j_1 + \cdots + j_k),$$

defined by

$$\gamma(g; f_1, \ldots, f_k) := g(f_1, \ldots, f_k).$$

For each j and each spectrum $E \in \mathscr{S}U^j$, the identity map on $\mathscr{L}(j)$ then induces a spectrum $\mathscr{L}(j) \ltimes E \in \mathscr{S}U$. (In the special case that $j = 0$, we in fact have that $\mathscr{L}(0) \ltimes X \cong \Sigma^\infty X$ for any based topological space X.) Moreover, the endofunctor

$$\mathscr{L}(1) \ltimes (-) : \mathscr{S}U \to \mathscr{S}U$$

defines a monad, which we denote by $\mathbb{L}$, with multiplication map $\mathbb{L}\mathbb{L}E \to \mathbb{L}E$ induced by the structure map $\gamma : \mathscr{L}(1) \times \mathscr{L}(1) \to \mathscr{L}(1)$, and unit $E \to \mathbb{L}E$ induced by the inclusion $\{0\} = \mathscr{L}(0) \to \mathscr{L}(1)$ which identifies id_U.

We write $\mathscr{S}[\mathbb{L}]$ for the category of $\mathbb{L}$-algebras, which we refer to as $\mathbb{L}$-*spectra*. This category is equipped with a bifunctor $\wedge_{\mathscr{L}} : \mathscr{S}[\mathbb{L}] \times \mathscr{S}[\mathbb{L}] \to \mathscr{S}[\mathbb{L}]$, called the *smash product of* $\mathbb{L}$-*spectra*, defined as follows: let M and N be $\mathbb{L}$-spectra, with actions $\xi_M : \mathbb{L}M \to M$ and $\xi_N : \mathbb{L}N \to N$; we take $M \wedge_{\mathscr{L}} N$ to be the coequalizer of the diagram

$$(\mathscr{L}(2) \times \mathscr{L}(1) \times \mathscr{L}(1)) \ltimes (M \wedge N) \underset{\gamma \ltimes \mathrm{id}}{\overset{\mathrm{id} \ltimes (\xi_M \wedge \xi_N)}{\rightrightarrows}} \mathscr{L}(2) \ltimes (M \wedge N).$$

Notation 5.13. Following the notation of [1], we denote the coequalizer of

$$(\mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_n) \times \mathscr{L}(k) \times \mathscr{L}(1)^k) \ltimes E \underset{(\mathrm{id} \times \gamma) \ltimes \mathrm{id}}{\overset{\mathrm{id} \ltimes \xi_E}{\rightrightarrows}} (\mathscr{L}(i_1) \times \cdots \times \mathscr{L}(i_n) \times \mathscr{L}(k)) \ltimes E$$

by

$$(\mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_n) \times \mathscr{L}(k)) \ltimes_{\mathscr{L}(1)^k} E.$$

As shown in Proposition I.5.2 and Theorem I.5.5 of [1] the smash product $\wedge_{\mathscr{L}}$ is symmetric and associative (up to natural isomorphism), and for each $M \in \mathscr{S}[\mathbb{L}]$ there is a canonical map

$$S \wedge_{\mathscr{L}} M \to M,$$

described in Proposition I.8.3 of [1].

An S-*module* is an $\mathbb{L}$-spectrum for which this map is an isomorphism, and we write $\mathscr{M}_S$ for the full subcategory of $\mathscr{S}[\mathbb{L}]$ spanned by the S-modules. For S-modules M and N, we write $M \wedge_S N := M \wedge_{\mathscr{L}} N$, which is again an S-module (see Proposition II.1.2 of [1]), so $\wedge_S$ defines a symmetric monoidal product on $\mathscr{M}_S$.

We aim to show that $\wedge_S$ is realized as the internalization of the external product of spectra $\wedge$. First, we must formulate a suitable graded monoidal category to impose an internalization upon. This is somewhat tricky, primarily because of the non-unitality of $\wedge_{\mathscr{L}}$. We essentially need to define a kind of intermediate, "non-unital" internalized category (i.e. we forget about the degree 0 component, and only require an associative product) which acts like $\mathscr{S}[\mathbb{L}]$ above, and then restrict to those objects which behave like S-modules to recover unitality.

Definition 5.14. Fix $m \geq 1$. Then there is a monad $\mathbb{L}^m$ on $\mathscr{S}U^m$, defined by

$$\mathbb{L}^m(E) := \mathscr{L}(1)^m \ltimes E,$$

where the map $\mathscr{L}(1)^m \to \mathscr{I}(U^m, U^m)$ carries a tuple of isometries $(f_1, \dots, f_m)$ to the isometry $f_1 \times \cdots \times f_m$. The multiplication map $\mu_m : \mathbb{L}^m\mathbb{L}^m E \to \mathbb{L}^m E$ is induced by m copies of the structure map $\gamma : \mathscr{L}(1) \times \mathscr{L}(1) \to \mathscr{L}(1)$, while the unit $\eta_m : E \to \mathbb{L}^m E$ is induced by the inclusion of $\mathscr{L}(0)$ into $\mathscr{L}(1)^m$ which identifies $(\mathrm{id}_U, \dots, \mathrm{id}_U)$.

The category $\mathscr{S}[\mathbb{L}^m]$ of $\mathbb{L}^m$-*spectra* is the category of algebras over $\mathbb{L}^m$.

Setting $\mathscr{S}[\mathbb{L}]_0 := \mathbf{Top}_*$, the categories $\mathscr{S}[\mathbb{L}]_m$ assemble to form a graded category $\mathscr{S}[\mathbb{L}]_\bullet$. The external smash product of spectra then defines an associative graded product on $\mathscr{S}[\mathbb{L}]_\bullet$.

Definition 5.15. For $m, n \geq 1$, there is a bifunctor $\mu : \mathscr{S}[\mathbb{L}]_m \times \mathscr{S}[\mathbb{L}]_n \to \mathscr{S}[\mathbb{L}]_{m+n}$, which, given $(M, \xi_M) \in \mathscr{S}[\mathbb{L}]_m$ and $(N, \xi_N) \in \mathscr{S}[\mathbb{L}]_n$, equips $M \wedge N$ with the structure map $\xi : \mathbb{L}^{m+n}(M \wedge N) \to M \wedge N$ defined by the composite

$$(\mathscr{L}(1)^m \times \mathscr{L}(1)^n) \ltimes (M \wedge N) \cong (\mathscr{L}(1)^m \ltimes M) \wedge (\mathscr{L}(1)^n \ltimes N) \xrightarrow{\xi_M \wedge \xi_N} M \wedge N.$$

The category $\mathscr{S}[\mathbb{L}]_\bullet$ will lay the foundation for our internalization of the smash product of spectra. However, before we can begin assembling this internalization, we will need a preliminary result, which generalizes Lemma I.5.4 of [1].

Lemma 5.16 (Hopkins)**.** For integers $k \geq 1$, $j_1, \dots, j_k \geq 1$, the diagram

$$\mathscr{L}(k) \times \mathscr{L}(1)^k \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \underset{\gamma \times \mathrm{id}}{\overset{\mathrm{id} \times \gamma^k}{\rightrightarrows}} \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \xrightarrow{\gamma} \mathscr{L}(j_1 + \cdots + j_k)$$

is a split coequalizer of spaces, and consequently

$$\mathscr{L}(j_1 + \cdots + j_k) \cong \mathscr{L}(k) \times_{\mathscr{L}(1)^k} \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k).$$

Proof. The argument is a direct generalization of the one presented in [1]. For each $i = 1, \dots, k$ we fix an isomorphism $s_i : U^{i_k} \cong U$. We define maps

$$\begin{aligned} h : \mathscr{L}(j_1 + \cdots + j_k) &\to \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \\ f &\mapsto (f\Big(\bigoplus_{i=1}^k s_i\Big)^{-1}; s_1, \dots, s_k) \end{aligned}$$

and

$$\begin{aligned} r : \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) &\to \mathscr{L}(k) \times \mathscr{L}(1)^k \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \\ (f; g_1, \dots, g_k) &\mapsto (f; g_1 s_1^{-1}, \dots, g_k s_k^{-1}; s_1, \dots, s_k), \end{aligned}$$

and claim that these split the fork above.

First, given $f \in \mathscr{L}(j_1 + \cdots + j_k)$, we observe that

$$\gamma h(f) = f\Big(\bigoplus_{i=1}^k s_i\Big)^{-1}(s_1, \dots, s_k) = f \bigoplus_{i=1}^k \mathrm{id}_{U^{j_i}} = f,$$

so γh is the identity. Additionally, given $(f; g_1, \dots, g_k) \in \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k)$, we have

$$\begin{aligned} (\mathrm{id} \times \gamma^k) r(f; g_1, \dots, g_k) &= (\mathrm{id} \times \gamma^k)(f; g_1 s_1^{-1}, \dots, g_k s_k^{-1}; s_1, \dots, s_k) \\ &= (f; g_1, \dots, g_k), \end{aligned}$$

so $(\mathrm{id} \times \gamma^k) r$ is the identity as well. Finally,

$$\begin{aligned} (\gamma \times \mathrm{id}) r(f; g_1, \dots, g_k) &= (\gamma \times \mathrm{id})(f; g_1 s_1^{-1}, \dots, g_k s_k^{-1}; s_1, \dots, s_k) \\ &= (f(g_1 s^{-1}, \dots, g_k s_k^{-1}); s_1, \dots, s_k), \end{aligned}$$

and

$$\begin{aligned} h\gamma(f; g_1, \ldots, g_k) &= h(f(g_1, \ldots, g_k)) \\ &= (f(g_1, \ldots, g_k)\Big(\bigoplus_{i=1}^{k} s_i\Big)^{-1}; s_1, \ldots, s_k) \\ &= (f(g_1 s^{-1}, \ldots, g_k s_k^{-1}); s_1, \ldots, s_k), \end{aligned}$$

so we conclude that $(\gamma \times \mathrm{id})r = h\gamma$. □

We now begin to construct the desired internalization $\Phi_\bullet$. It may seem strange to do this before defining the graded monoidal category that we ultimately aim to internalize, but this category will in fact be defined using our "incomplete" internalization of $\mathscr{S}[\mathbb{L}]_\bullet$.

Definition 5.17. Given $M \in \mathscr{S}[\mathbb{L}]_m$, we define the underlying object of $\Phi_m(M, \xi_M)$ to be the coequalizer of the diagram

$$(\mathscr{L}(m) \times \mathscr{L}(1)^m) \ltimes M \underset{\gamma \ltimes \mathrm{id}}{\overset{\mathrm{id} \ltimes \xi_M}{\rightrightarrows}} \mathscr{L}(m) \ltimes M.$$

The structure map $\mathbb{L}\Phi_m(M) \to \Phi_m(M)$ is induced by $\gamma : \mathscr{L}(1) \times \mathscr{L}(m) \to \mathscr{L}(m)$.

We would hope that the collection of functors Φ_m would assemble to form an internalization Φ on the graded monoidal category $(\mathscr{S}[\mathbb{L}]_\bullet, \mu, S^0)$. Indeed, this is very nearly the case – in all higher (i.e. nonzero) degrees, these functors do in fact enjoy both left and right absorbers.

Lemma 5.18. Let $m, n > 0$, let $M \in \mathscr{M}_m$, and let $N \in \mathscr{M}_n$. Then there exist natural isomorphisms

$$\Phi_{n+1}\mu(\Phi_m M, N) \cong \Phi_{m+n}\mu(M, N) \cong \Phi_{m+1}\mu(M, \Phi_n N)$$

Proof. We will show explicitly only that the left isomorphism exists, as the right isomorphism follows from a symmetric argument. Since N is an $\mathbb{L}^n$-algebra there is a split coequalizer diagram

$$\mathbb{L}^n\mathbb{L}^n N \rightrightarrows \mathbb{L}^n N \longrightarrow N,$$

so $N \cong \mathscr{L}(1)^n \ltimes_{\mathscr{L}(1)^n} N$. Thus, we have

$$\begin{aligned} \Phi_{n+1}\mu(\Phi_m M, N) &= \mathscr{L}(n+1) \ltimes_{\mathscr{L}(1)^{n+1}} (\mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} M \wedge N) \\ &\cong \mathscr{L}(n+1) \ltimes_{\mathscr{L}(1)^{n+1}} (\mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} M \wedge \mathscr{L}(1)^n \ltimes_{\mathscr{L}(1)^n} N) \\ &\cong \mathscr{L}(n+1) \ltimes_{\mathscr{L}(1)^{n+1}} ((\mathscr{L}(m) \times \mathscr{L}(1)^n) \ltimes_{\mathscr{L}(1)^{m+n}} (M \wedge N)). \end{aligned}$$

But by Lemma 5.16 we have

$$\begin{aligned} \mathscr{L}(m+n) \ltimes_{\mathscr{L}(1)^{m+n}} (M \wedge N) &\cong (\mathscr{L}(n+1) \times_{\mathscr{L}(1)^{n+1}} \mathscr{L}(m) \times \mathscr{L}(1)^n) \ltimes_{\mathscr{L}(1)^{m+n}} (M \wedge N) \\ &\cong \mathscr{L}(n+1) \ltimes_{\mathscr{L}(1)^{n+1}} ((\mathscr{L}(m) \times \mathscr{L}(1)^n) \ltimes_{\mathscr{L}(1)^{m+n}} (M \wedge N)), \end{aligned}$$

so we conclude that

$$\Phi_{n+1}\mu(\Phi_m M, N) \cong \mathscr{L}(m+n) \ltimes_{\mathscr{L}(1)^{m+n}} (M \wedge N) = \Phi_{m+n}\mu(M, N),$$

which is the desired isomorphism. □

A priori the internalized product $\wedge_{\mathscr{L}} := \Phi_2\mu_{1,1}$ of $\mathbb{L}$-spectra is associative (see Theorem I.5.5 of [1]). One would hope that S would be the unit of this product, but sadly this is not the case. (The problem, as hinted at above, is the absence of absorbers in degree 0.) As a consolation, we do, however, get a canonical map

$$S \wedge_{\mathscr{L}} M \to M$$

for any $\mathbb{L}$-spectrum M, described in Proposition I.8.3 of [1]. In fact, this generalizes to higher degree spectra and to spaces besides S^0:

Lemma 5.19. Let (M, ξ_M) be an $\mathbb{L}^m$-spectrum, and let X be a based topological space. Then there is a natural morphism of $\mathbb{L}$-spectra

$$\Phi_{m+1}\mu(\Phi_0 X, M) \to \Phi_m\mu(X, M).$$

Proof. First, suppose M is a free $\mathbb{L}^m$-algebra $\mathbb{L}^m E = \mathscr{L}(1)^m \ltimes E$, where $E \in \mathscr{S}[U^m]$. Then we seek a natural morphism

$$\mathscr{L}(m+1) \ltimes_{\mathscr{L}(1)^{m+1}} (\Sigma^\infty X \wedge \mathscr{L}(1)^m \ltimes E) \to \mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} (X \wedge \mathscr{L}(1)^m \ltimes E).$$

First, we observe that γ defines a morphism

$$\mathscr{L}(m+1) \ltimes_{\mathscr{L}(1)^{m+1}} \mathscr{L}(0) \times \mathscr{L}(1)^m \to \mathscr{L}(m),$$

since it coequalizes the pair of morphisms

$$\mathscr{L}(m+1) \times \mathscr{L}(1)^{m+1} \times \mathscr{L}(0) \times \mathscr{L}(1)^m \underset{\gamma\times\mathrm{id}}{\overset{\mathrm{id}\times\gamma^{m+1}}{\rightrightarrows}} \mathscr{L}(m+1) \times \mathscr{L}(0) \times \mathscr{L}(1)^m .$$

This yields a map from

$$\begin{aligned}\mathscr{L}(m+1) \ltimes_{\mathscr{L}(1)^{m+1}} (\Sigma^\infty X \wedge \mathscr{L}(1)^m E) &= \mathscr{L}(m+1) \ltimes_{\mathscr{L}(1)^{m+1}} (\mathscr{L}(0) \ltimes X \wedge \mathscr{L}(1)^m \ltimes E)\\ &\cong (\mathscr{L}(m+1) \times_{\mathscr{L}(1)^{m+1}} \mathscr{L}(0) \times \mathscr{L}(1)^m) \ltimes (X \wedge E)\end{aligned}$$

to

$$\mathscr{L}(m) \ltimes (X \wedge E).$$

But by Lemma 5.16 we have

$$\mathscr{L}(m) \cong \mathscr{L}(m) \times_{\mathscr{L}(1)^m} \mathscr{L}(1)^m,$$

so

$$\begin{aligned}\mathscr{L}(m) \ltimes (X \wedge E) &\cong \mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} (\mathscr{L}(1)^m \ltimes (X \wedge E))\\ &\cong \mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} (X \wedge (\mathscr{L}(1)^m \ltimes E)).\end{aligned}$$

Composing the morphisms above yields the desired map.

Now, if M is any $\mathbb{L}^m$-spectrum then there is a split coequalizer diagram

$$\mathbb{L}^m\mathbb{L}^m M \rightrightarrows \mathbb{L}^m M \twoheadrightarrow M .$$

Since split coequalizers are absolute colimits, the canonical morphisms corresponding to the free algebras in the diagram induce a morphism of coequalizer diagrams

$$\begin{tikzcd}
\Phi_{m+1}\mu(\Phi_0 X, \mathbb{L}^m\mathbb{L}^m M) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \Phi_{m+1}\mu(\Phi_0 X, \mathbb{L}^m M) \arrow[r, two heads] \arrow[d] & \Phi_{m+1}\mu(\Phi_0 X, M) \arrow[d, dashed] \\
\Phi_m\mu(X, \mathbb{L}^m\mathbb{L}^m M) \arrow[r, shift left] \arrow[r, shift right] & \Phi_m\mu(X, \mathbb{L}^m M) \arrow[r, two heads] & \Phi_m\mu(X, M)
\end{tikzcd}$$

which induces the desired map for M. $\square$

In general, the map in Lemma 5.19 need not be an isomorphism. To obtain the desired internalized category, we restrict our domain to only those $\mathbb{L}_m$-spectra for which it is.

Definition 5.20. Given $m \geq 1$, let $\mathscr{M}_m$, *the category of S-modules of degree m*, be the full subcategory of $\mathscr{S}[\mathbb{L}]_m$ spanned by those $\mathbb{L}^m$-algebras M such that the canonical map

$$\Phi_{m+1}\mu(\Phi_0 X, M) \to \Phi_m \mu(X, M)$$

of Lemma 5.19 an isomorphism.

Additionally, let $\mathscr{M}_0 := \mathbf{Top}_*$, the category of based spaces, and let $\Phi_0 := \mathscr{L}(0) \ltimes (-) = \Sigma^\infty$ (which, by Proposition II.1.2 of [1], defines a functor from based spaces to S-modules). Then, by way of the following proposition, we can naturally extend μ to a graded monoidal product on $\mathscr{M}_\bullet$, with unit S^0.

Proposition 5.21. Let $M \in \mathscr{M}_m$, and let $N \in \mathscr{M}_n$. Then $\mu(M, N) \in \mathscr{M}_{m+n}$.

Proof. Let X be a based topological space. We aim to show that the map

$$\Phi_{m+n+1}\mu(\Phi_0 X, \mu(M, N)) \to \Phi_{m+n}\mu(X, \mu(M, N))$$

is an isomorphism.

First, suppose $m, n > 0$. From the associativity of μ, we have

$$\Phi_{m+n+1}\mu(\Phi_0 X, \mu(M, N)) \cong \Phi_{m+n+1}\mu(\mu(\Phi_0 X, M), N).$$

Now, since $\Phi_0 X$ and $\mu(M, N)$ both have positive degree, we can apply Lemma 5.18 to conclude that

$$\Phi_{m+n+1}\mu(\mu(\Phi_0 X, M), N) \cong \Phi_{n+1}\mu(\Phi_{m+1}\mu(\Phi_0 X, M), N).$$

But M is a degree m S-module, so we have

$$\Phi_{n+1}\mu(\Phi_{m+1}\mu(\Phi_0 X, M), N) \cong \Phi_{n+1}\mu(\Phi_m \mu(X, M), N).$$

Since N has positive degree, we can again utilize Lemma 5.18, from which we obtain

$$\Phi_{n+1}\mu(\Phi_m \mu(X, M), N) \cong \Phi_{m+n}\mu(\mu(X, M), N).$$

Finally, we again use the associativity of μ to conclude that

$$\Phi_{m+n+1}\mu(\Phi_0 X, \mu(M, N)) \cong \Phi_{m+n}\mu(X, \mu(M, N)),$$

as desired.

Next, suppose (without loss of generality) that $m = 0$ (i.e. that M is a based topological space) and $n > 0$. (The case when $m > 0$ and $n = 0$ follows from the symmetry of μ.) Using the associativity of μ we have

$$\Phi_{n+1}\mu(\Phi_0 X, \mu(M, N)) \cong \Phi_{n+1}\mu(\mu(\Phi_0 X, M), N).$$

Moreover, we have

$$\mu(\Phi_0 X, M) = (\Sigma^\infty X) \wedge M \cong (\mathscr{L}(0) \ltimes X) \wedge M,$$

and by properties of the twisted half-smash product it follows that

$$\mu(\Phi_0 X, M) \cong \mathscr{L}(0) \ltimes (X \wedge M) \cong \Sigma^\infty(X \wedge M) = \Phi_0 \mu(X, M).$$

Therefore, we have

$$\Phi_{n+1}\mu(\mu(\Phi_0 X, M), N) \cong \Phi_{n+1}\mu(\Phi_0 \mu(X, M), N),$$

and since N is a degree n S-module

$$\Phi_{n+1}\mu(\Phi_0 \mu(X, M), N) \cong \Phi_n(\mu(X, M), N).$$

Applying the associativity of μ again yields the isomorphism

$$\Phi_{n+1}\mu(\Phi_0 X, \mu(M,N)) \cong \Phi_n(X, \mu(M,N)).$$

Lastly, if both m and n are 0 then M and N are based topological spaces, in which case so is $\mu(M,N)$. Thus $\mu(M,N)$ is likewise an S-module of degree 0, completing the proof. $\square$

It follows that μ restricts to a graded monoidal product on $\mathscr{M}_\bullet$.

Remark 5.22. Note that $\mathscr{M}_1 = \mathscr{M}_S$, the category of S-modules (see Definition II.1.1 of [1]). Certainly each spectrum in $\mathscr{M}_1$ is an S-module, by taking $X = S^0$. Conversely, every classical S-module is a degree 1 S-module by Proposition II.1.4 of [1].

Remark 5.23. If $M \in \mathscr{M}_m$ then $\Phi_m(M) \in \mathscr{M}_1$, so Φ_m restricts to a functor $\mathscr{M}_m \to \mathscr{M}_1$. This is because, given a based space X, we have isomorphisms

$$\Phi_2\mu(\Phi_0 X, \Phi_m M) \cong \Phi_{m+1}\mu(\Phi_0 X, M) \cong \Phi_m\mu(X, M),$$

where the first map is the isomorphism described in Lemma 5.18, and the second follows from the fact that M a degree m S-module.

We can now verify that the functors $\Phi_\bullet$ indeed define an internalization of μ. For this, we will need a preliminary results, which generalizes Lemma I.5.4 of [1].

Theorem 5.24. The functors Φ define an internalization on the graded monoidal category $(\mathscr{M}_\bullet, \mu, S^0)$, and the internalized product on $\mathscr{M}_S$ is precisely $\wedge_S$.

Proof. We aim to construct absorbers for Φ, as defined in Definition 3.18. We will content ourselves with identifying the left absorbers, as the right absorbers are built through an identical procedure, and omit the verification of the coherence axioms.

Let $M \in \mathscr{M}_m$, $N \in \mathscr{M}_n$. We seek a natural isomorphism

$$\Phi_{n+1}\mu(\Phi_m M, N) \cong \Phi_{m+n}\mu(M,N).$$

First, if $m, n > 0$ then the left (and right) absorbers are given by restricting the isomorphisms of Lemma 5.18 to S-modules.

Next, suppose $m = 0$. Then M is a based topological space, and since N is a degree n S-module we have a natural isomorphism

$$\Phi_{n+1}\mu(\Phi_0 M, N) \cong \Phi_n\mu(M,N)$$

by Definition 5.20.

Finally, suppose $n = 0$. Then N is a based topological space, and thus we have natural isomorphisms

$$\begin{aligned}\Phi_1\mu(\Phi_m M, N) &= \mathscr{L}(1) \ltimes_{\mathscr{L}(1)} ((\mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} M) \wedge N) \\ &\cong (\mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} M) \wedge N \\ &\cong \mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} (M \wedge N) \\ &= \Phi_m\mu(M,N),\end{aligned}$$

completing the proof. $\square$

Having shown that the smash product $\wedge_S$ of S-modules is an internalization of $\wedge$, we can immediately conclude that it is symmetric, using the formal properties of Φ.

Lemma 5.25. $(\mathscr{M}_\bullet, \mu, S^0)$ is a symmetric graded monoidal category, and Φ is a symmetric internalization.

Proof. The symmetry of $(\mathscr{M}_\bullet, \mu, S^0)$ comes from that of the external smash product $\wedge$. Naturality of the isomorphisms which define the left and right absorbers in the proof of Theorem 5.24 shows that Φ is symmetric. $\square$

Proposition 5.26. The monoidal product $\wedge_S$ on the category of S-modules $\mathscr{M}_S$ is symmetric.

Proof. This follows immediately from Lemma 5.25 and Corollary 3.29. $\square$

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